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THE LAW OF THE GEOIDAL SLOPE AND FALLACIES IN DYNAMIC METEOROLOGY.

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CONTENTS.

SYNOPSIS.

The action of gravity upon bodies moving over a rotating globe is expressed in two wholly independent inertia reactions. One of these has long been known and dignified by a specific title, *the law of equal areas*. Its action in the dynamics of the atmosphere has been fully discussed, even exploited, by practically all writers on the subject. The other reaction has also long been known, but strangely enough has never been christened. Nameless and neglected, the important part it plays in controlling the motions of the air has been overlooked and misunderstood, or even ascribed to friction and other actions, with the result that serious fallacies have been introduced in many textbooks and writings both by the popular authors and even the mathematicians.

The present paper aims to clear away these mistakes and proposes that the neglected principle be dignified by the name of *the law of the geoidal slope*.

The two actions are inseparable, simultaneous in their operation, not directly antagonistic, but coordinate and complementary. Their action on a body in frictionless motion on a rotating globe is analyzed and made clear.

The general motions of the atmosphere are briefly discussed as steady motions under forces balanced against resistances, and the principal equations of motion are given for cyclonic, anticyclonic, and parallel systems of isobars, including a table of gradient winds for different latitudes and conditions.

Numerous quotations from both popular and mathematical writers are submitted, accompanied by notes clearly showing the errors herein claimed to exist.

The crucial question at issue is submitted to be: What is the nature of the frictionless circulation of the air of a polar hemisphere assumed to be warm at the equator and cold at the poles? A rational answer to this question is offered in which the irreconcilable differences between the frictionless polar cyclone of the mathematicians and a rational polar cyclone derived from the equations of the gradient winds are clearly exhibited.

The basic conditions which underlie the general circulation of the atmosphere are stated in 12 fundamental principles.

HISTORICAL.

Many standard textbooks on Meteorology when treating of the motions of the atmosphere call attention to the superhurricane winds which are called for by the operation of the law of equal areas and claim these inconceivable velocities are prevented in nature by atmospheric friction, convection, turbulence, etc.

It appears W. M. Davis was the first to call attention to the fallacy involved in such representations (Elementary Meteorology, p. 103), which he explained by stating that the deflective influence of the earth's rotation changes the direction *only* of a freely moving body and can' produce no effect on the velocity. He adds, "If a body were given a velocity of 25 miles an hour to the south when in latitude 30 N. and was supposed to move without friction over a level surface, it would continue to move at the same moderate rate whatever latitude it reached."

As statements of facts, Davis's representations are perfectly correct, but as an explanation of a fallacy they are insufficient, because the operation of the law of equal areas is one thing, while the action of the deflective influence of the earth's rotation on bodies in free frictionless motion is an entirely separate and different thing. These two influences are just as separate and different in their characteristics and actions, to use an imperfect analogy, as water is different from one of its constituents, oxygen or hydrogen.

In the present state of this subject no valid representations appear to have yet been made to show just why it is the actual motions of the atmosphere are gentle and beneficent whereas the theory of equal areas, which can not be questioned, calls for winds of superhurricane force and inconceivable velocities.

Mr. H. W. Clough, in an article in this Review (Aug., 1920, 48: 463), enlarges upon Davis's explanation of this friction fallacy, and explains it by resort to the deflective influence of the earth's rotation. These efforts, however, do not constitute a sufficient explanation of the fallacy or serve to account for the errors in the mathematical writings of Ferrel, Bigelow, and others.

Any careful reader of either Davis's or Clough's remarks on the subject must wonder what discrimination should be made between the law of equal areas and the deflective influence of the earth's rotation, or he must infer that both Davis and Clough regarded these two concepts as more or less identical. In fact, the whole fallacy has arisen and spread chiefly because writers of high authority have failed to adequately recognize the distinction and the relation.

With full acknowledgment to the authors mentioned, the writer believes the present paper sets forth and offers the first correct explanation of serious fundamental errors which have appeared in nearly all meteorological writings, both popular and mathematical, during the past 60 years.

In order to present our subject matter in a direct and logical manner, it will be necessary to review in the briefest possible way what is already well known, but not always consistently applied relative to—(1) the law of the constancy of momentum, (2) the properties of a gioidal surface, (3) the deflective influence of the earth's rotation, its effects on bodies in frictionless motion and its relations to (1) and (2).

Readers who are already familiar with the fundamental principles of dynamics governing the motions of the atmosphere may pass over Sections I to V, inclusive, and read at once the criticisms in Section VI.

I. THE LAW OF THE CONSTANCY OF MOMENTUM.

This basic law of matter is of universal application and its demands must of course be satisfied in all cases. Nevertheless, great care must be observed as to how the law is applied to the motions of the atmosphere and of bodies moving freely over any rotating globe.

The law simply states the fact of nature that unless acted upon by extraneous forces the *momentum* of a given mass in motion remains constant. If an extraneous central force causes the body to move about a point or axis then the *moment of momentum* remains constant.

Applied to motions on the earth's surface or any rotating globe the equation, stating this law, may be written:

$$Er = E_1 r_1 \text{ or } E \cos \varphi = E_1 \cos \varphi_1 = \frac{\text{constant}}{R} \quad (1)$$

in which r and r_1 are the respective distances of the body from the axis of rotation on a globe with mean radius R. E_1 then becomes the component of velocity in longitude (eastward or westward) of a body on reaching the latitude φ_1 after leaving the latitude φ where its motion in longitude was E.

Equation (1) suggests the origin of the name "the law of equal areas," because for elemental motions the products like Er in any case of motion around a point is twice the area swept over in a small unit of time by the vector r, and in the case of rotation about an axis the product $E \cos \varphi$ for elementary motions represents, for unit radius, twice the area swept over by the vector from the moving body to the axis of rotation *when said area is projected on the plane of the equator*. By equation (1) these areas are equal, each to each, whence the name.

It will be noticed equation (1) is wholly independent of the velocity by which the change of latitude occurs. A given change of latitude may occur in a minute or a year and cause exactly the same change in the eastward (or westward) velocity of the body if no other forces are in action.

The rotative velocity of a particle at the earth's surface in miles per hour at any latitude φ is

$$E = 1038.7 \cos \varphi \quad (2)$$

The table below gives values computed from equations (1) and (2) for a body assumed to move without friction exactly poleward from rest at the equator.

TABLE 1.—*Velocities on the rotating earth satisfying the law of equal areas.*

[Velocities in miles per hour.]

Latitude.	Eastward surface velocity. E.	Eastward velocity of body on arrival.	Eastward velocity of body over ground.	Ferrel[1] (+ eastward, − westward).
	2	3	4	5
° ′				
0 00	1,039	1,039	0	− 346
10 00	1,023	1,055	32	− 320
20 00	976	1,105	129	− 239
30 00	900	1,199	299	− 100
35 16	848	1,115	377	0
40 00	796	1,356	560	+ 108
50 00	668	1,616	948	+ 410
60 00	519	2,077	1,558	+ [illegible]65
70 00	355	3,037	2,682	+1,669
80 00	180	5,981	5,801	+3,807
90 00	0	∞	∞	+ ∞

[1] Velocities satisfying Ferrel's equation for frictionless circumpolar cyclone with maximum pressure at latitude 35° 16′. See discussion in later section on citations from mathematicians.

Many textbooks represent that the impossible velocities in column 4 would occur in nature if it were not for friction and various internal wastes of kinetic energy. These teachings are erroneous, because bodies can not be moved over the earth's surface subject *only* to the law of equal areas. The computations above are applicable to a body on the earth only on the assumption that the downward pull of gravity *as it acts on the moving body* is exactly perpendicular to the smooth geoid. This can not be the case, and herein lies the very root of the fallacy.

The popular writers especially seem to have exploited the operation of the law of equal areas and emphasized the superhurricane velocities changes of latitude would require if not restrained by friction. In such teachings the operation of what we now call the law of the geoidal slope has not been adequately recognized. This is the more surprising because it is also clearly set out in many, sometimes even in the same textbooks and writings, that bodies in frictionless motion over a rotating globe follow a curved path *without change of velocity*. Davis says: "A body once set in motion under these conditions would continue moving forever, always changing its direction but never its velocity." [2]

All the facts of the matter were made clear by Ferrel in 1858, but they have not been consistently applied in many cases, even by Ferrel himself, and therefore emphasis must be placed upon the neglected details, the principle of the geoidal slope.

II. THE LAW OF THE GEOIDAL SLOPE.

It has long been taught and recognized that the figure of the earth is not truly spherical, but that because of the revolution about its axis the form is geoidal, which by definition ~~is a form~~ the surface of which at each point is perpendicular to the plumb line at that place. This condition would be fully satisfied if the earth's surface were entirely of water or other liquid.

General statement of the law.—The properties of a geoidal surface, assumed to rotate from the west to the east, may be comprehended in a single statement as follows:

A geoidal surface is a neutral or horizontal surface only for bodies at rest upon it. That is, gravity is powerless to set up any lateral motions among such bodies. The surface slopes toward the equator for every body having a relative motion eastward and toward the pole for every body with a motion westward. A component of the force of gravity pulls the moving bodies down the slopes.

This principle follows directly from the action of gravity on a rotating yielding globe. Assuming homogeneity, the figure of equilibrium will be spherical if the globe is at rest. If rotating about an axis through the center of mass the centrifugal reaction gives rise to a component of gravity which acts tangent to the surface and toward the equator. This force causes flattening at the poles and bulging toward the equator. The amounts will be nicely adjusted to the speed of rotation, and for equilibrium the resultant downward pull of gravity, represented in direction by the plumb line, will be just perpendicular to the surface at each place.

Assuming that the globe revolves from the west to the east, then any body which moves eastward over the surface will actually move more rapidly around the axis

[2] Davis, Wm. M.: Elementary Meteorology, p. 104. See also Sprung: On the paths of particles moving freely on the rotating surface of the earth, etc.; English translation by Abbe and Russel, Smith. Misc. Coll., vol. 51, No. 4. Whipple: The motion of a particle on a smooth rotating globe, Phil. Mag., vol. 33, 6th series, 1917, p. 457.

than the geoid itself, and for this body the equator is not bulged out enough—that is, the geoid slopes downward toward the equator. Just the reverse is true if the body moves to the westward, because it is then revolving more slowly than the geoid about the axis, and the equator is then bulged too much.

It is highly important in the study of the general circulation of the atmosphere that the student form a clear mental picture of the real terrestrial conditions brought about by the operations of this geoidal law.

For all the winds of the globe moving eastwardly, the surface of the earth is like a trough with its axis or bottom line coinciding with the equator and its lateral slopes rising higher and more steeply with latitude and the eastward velocity. A component of the force of

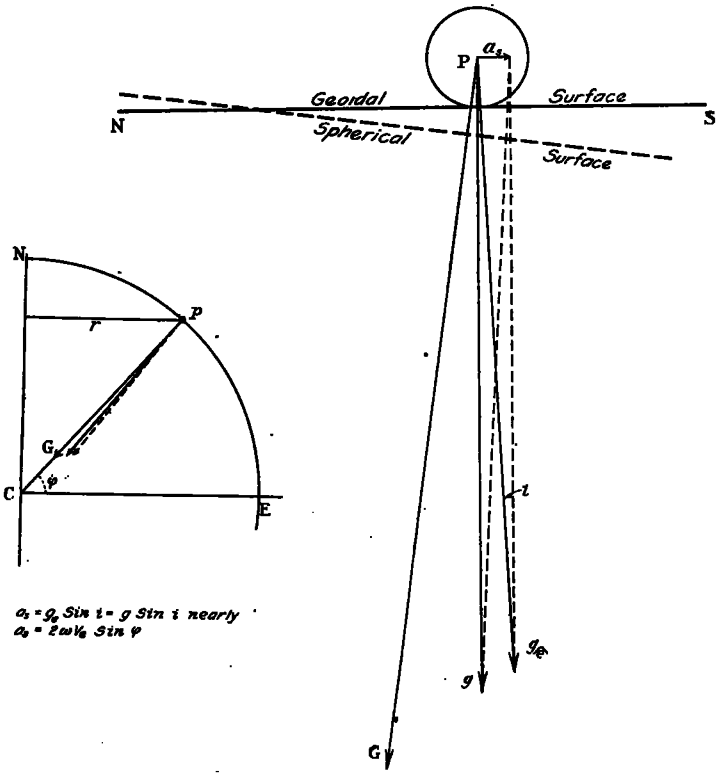

FIG. 1.—Action of gravity on matter on a rotating globe. For globe at rest, surface is spherical and plumb line extended passes through center. In rotation, surface is geoidal, plumb line Pg passes beside center, and force is smaller. If body moves eastward, plumb line inclined farther from center Pg_e and component a_s pulls body down geoidal slope.

$$a_s = g_e \sin i = g \sin i \text{ nearly} = 2\omega\ V_e \sin\varphi$$

gravity continually drives such winds down the slope toward the equator.

On the other hand, the equator is like a ridge or a geoidal divide for all winds moving to the westward. The hemispheres in this case are bowl-like forms to westwardly moving masses of air which are therefore urged poleward by a component of gravity acting down the slopes.

Students who are even fairly acquainted with meterorological literature, will recognize that there is little if anything essentially new in the fundamental principles involved in the foregoing statements.[3] These truths have long been known but the novelty of the present effort is the emphasis laid upon the fact that the surface of the earth in general is an up-hill surface in one sense to every moving body upon it, except when the motion is exactly along a meridian or the equator. There is in action at all times on such bodies, therefore, a component of gravity drawing them toward the equator or the poles according to the latitude and the relative velocity in longitude. The relation of this matter to the operation of the law of equal areas as applied to the motions of the atmosphere in latitude, has long been seriously neglected or overlooked.

It is the pull of gravity down the geoidal slopes, not friction, which prevents the superhurricane velocities exploited in the textbooks.

With the foregoing brief discussion of (1) the law of equal areas and (2) the law of the geoidal slope, it will be easy to understand the combination of their effects into the well-known deflective influence of earth's rotation.[4]

III. THE DEFLECTIVE INFLUENCE OF THE EARTH'S ROTATION.

It is very suggestive of the profound obscurity of this subject to recognize that it has occupied the attention of scientists for fully 200 years; nevertheless several of the most recent writings contain erroneous statements concerning its application in both meteorology and astronomy.

Two rotation effects.—It must not be forgotten, moreover, that more than 60 years ago Wm. Ferrel[5] without emphasizing in any way the parts played separately by the two components was, however, the first fully to analyze and evaluate both influences of rotation as two wholly separate and independent inertia reactions. The operation of these require (*a*) that changes in latitude must be accompanied by accelerations in longitude to satisfy the demands of the law of equal areas; and (*b*) that changes in longitude must be accompanied by forces in the meridians arising from a tangential component of gravity which it is now proposed to comprehend under the name of *the law of the geoidal slope.*

Ferrel also showed that the resultant of these two coordinate and simultaneous influences was entirely passive in its character, because it always acted exactly perpendicular to the line of motion of any body and therefore could not change the velocity but always changed the direction of motion.

The value of the force on a mass of m grams at latitude φ, moving in any direction at a velocity V centimeters per second is:

$$f = 2\omega m\, V \sin\varphi \text{ ---- Earth's deflective influence} \quad (3)$$

in which $\omega = \frac{2\pi}{86164}$ = angular velocity per second of the rotation of the earth on its axis.

Since the possible changes in latitude, φ, are small for ordinary values of the velocity, V, along the path the term $\sin\varphi$ is practically constant for any one locality, hence f is proportional to V. When this is the only deflecting force in operation the radius of curvature of the path is given by the equation:

$$r = \frac{V}{2\omega \sin\varphi} \quad (4)$$

and r is also constant for any one locality, and therefore the path of the body (if not too close to the equator) will be nearly a circle not quite closed on its western side.

[3] Ferrel: Professional Papers, Sign. Serv. No. 8, p. 48, sec. 100: Popular Treatise on Winds, p. 110, sec. 76. Davis: Elementary Meteorology, footnote, p. 115.

[4] For a full explanation, see MONTHLY WEATHER REVIEW, September, 1916, 43, 506.

[5] Ferrel, Wm.: Professional Papers No. 8, Signal Service, Annual Report, Chief Signal Officer, 1885, pt. 2, A Popular Treatise on Winds, ch. 2.

This path will be traversed at a uniform velocity and the time of rotation is easily seen to be:

$$T=\frac{2\pi r}{V}=\frac{\pi}{\omega \sin \varphi}$$

which is wholly independent of the velocity of the body. If T is measured in siderial hours $\omega=\frac{2\pi}{24}$ and $T=\frac{12}{\sin \varphi}$. Near the poles, where $\sin \varphi$ is sensibly unity, the body completes the circuit of its orbit in the time of half a rotation of the earth. In fact, at any latitude the angular change in the direction of motion of any freely moving body is double the angular turning of the ground by the earth's rotation.

The parts played by the two components of the deflective influence are clearly brought out in the discussion of figure 2.

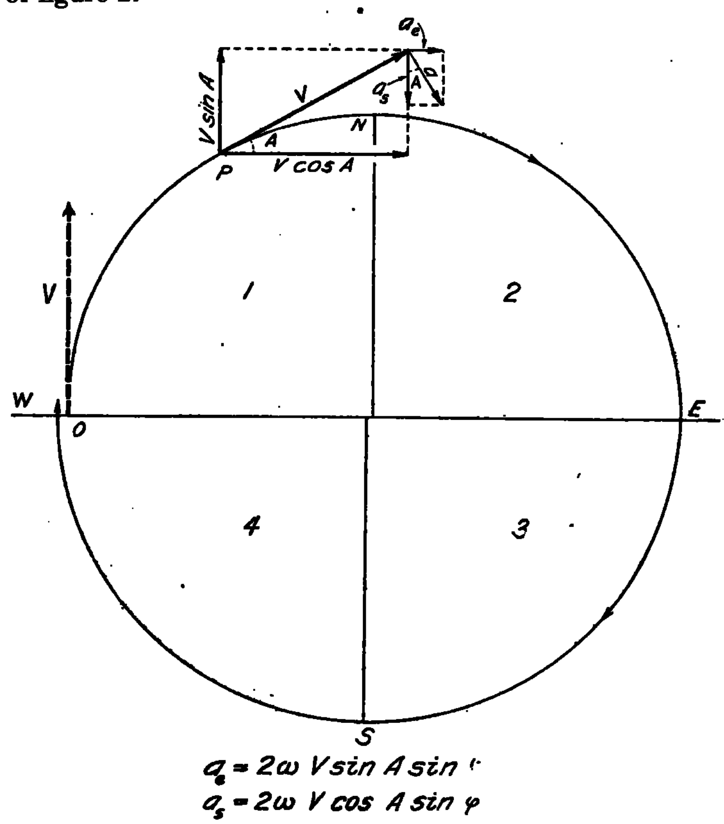

FIG. 2.—Path of body in frictionless motion at moderate velocity in Northern Hemisphere.

The case of frictionless motion.

At P the body will be moving at an angle A to the west-east line. The north component, $V \sin A$, and the east component, $V \cos A$, will give rise respectively to the accelerations eastward—

$a_e = 2\ \omega\ V \sin A \sin \varphi$ - - - - angular momentum effect and southward,

$a_s = 2\ \omega\ V \cos A \sin \varphi$ - - - - slope effect.

This southward acceleration a_s (or northward as the case may require) is the extraneous force which prevents the free operation of the law of equal areas. *Without* the slope effect, or acceleration a_s, figure 2, any body set moving northward in a frictionless manner would continue to move northward indefinitely. If started at the equator it would acquire the enormous velocities eastward given in Table 1, column 4. *With* the acceleration a_s the northward velocity is continuously checked until reduced to zero. Practically all writers on this subject up to the present time have disregarded the important part the acceleration a_s plays in terrestrial motions and that a_s and a_e are complementary and inseparable in their action. *Furthermore, the law of the conservation of momentum is satisfied at every point of the motion but without any change in the velocity of the body over the ground.*

Resuming the explanation of figure 2 and tracing the action of the accelerations a_e and a_s through the four quadrants of motion, it is plain that the momentum effect (the law of equal areas) has, at P given the body the eastward velocity $V \cos A$. At the same time the slope effect has reduced the initial northward velocity from V to $V \sin A$, and will presently completely destroy *all* the northward velocity. This is not lost, however, but has been transformed to eastward velocity by the law of equal areas. In fact, the *eastward or westward velocity at every point of the orbit is exactly that appropriate to the latitude and the initial velocity.* The law of equal areas accomplishes this control. If it were not for the slope effect the law of equal areas would produce indefinitely greater and greater velocities with each successive latitude reached.

When the body has reached its highest latitude northward it will be running due east on a southward sloping hillside and therefore be subject to a downhill component of gravity the amount of which is given by the equation:

$$g \sin \alpha = a_s = 2\ \omega\ V_e \sin \varphi$$

whence the inclination of the geoidal slope can be obtained from the equation:

$$\sin \alpha=\frac{2\ \omega\ V_e \sin \varphi}{g} \qquad (7)$$

in which V_e is the eastward or westward velocity as the case may require.

As soon as the body in the second quadrant runs downhill (southward) under the slope effect the momentum effect cuts down and presently completely destroys *all* eastward velocity. The body has now returned to the latitude from which it started, and the slope effect has given it the maximum velocity southward.

In the third quadrant the momentum effect must give accelerated *westward* velocities and the body finds itself running uphill on a geoidal slope, only to come to rest at the highest point it can attain (the lowest latitude), thence turning it runs down the slope in the fourth quadrant of its orbit, which returns to a point a little westward of the starting point as indicated. This hiatus is due to the variations in $\sin \varphi$ along the path and the corresponding changes in curvature.

From this analysis it is clearly seen that in general every body moving freely and in the most frictionless manner conceivable is nevertheless constantly controlled by the two contending influences, the latitude effect and the slope effect. The law of equal areas is perfectly valid, and if acting alone and in so far as changes of latitude are concerned, would create just such velocities in longitude as have been ascribed to it. However, no such influence is free to act, or, more correctly, Nature opposes and nullifies one accelerating cause by another, the one the well-known law of equal areas; the other the law of the geoidal slope. The two are inseparably associated, simultaneous in their operation, not antagonistic of each other, but coordinate and complementary.

Thus it is seen there is no need to invoke friction and other wastes of energy to explain away the incredible velocities in longitude due to the operation of the law of equal areas. These velocities are automatically controlled by gravity itself acting directly upon the masses in motion as they run upon the geoidal slopes created by the motions and instantly adjusted in steepness to the requirements of the case by the momentary velocities in longitude.

Thus far we have considered only frictionless motions at a uniform velocity due to an initial impulse. We must next consider in the briefest possible manner the steady motions of the air under constantly acting pressure gradients with or without friction. We need not consider how the gradients or the resistances are produced or maintained, but the results attained will aid in readily understanding the fallacies to be shown in the quotations which will be given later.

IV. STEADY MOTIONS UNDER FORCES BALANCED AGAINST RESISTANCES.

Pressure gradients.—The immediate forces which produce the general motions of the atmosphere arise by virtue of, and are measured by, pressure gradients. Such gradients in air and other fluids are gravity reactions and in the grand case of the whole atmosphere the permanent gradients depend chiefly upon the *great contrasts of temperature* which are perpetually maintained by the unequal heating of the earth's surface by the sun, which, in conjunction with the continuous loss of heat by terrestrial radiation, cause the perpetual warmth of the Tropics and the extreme cold of the polar regions.

Resistances.—Convections of all kinds, turbulence, and eddy motions probably constitute by far the greatest resistances to atmospheric motions, but these are augmented by surface frictions and the flow over and around obstacles, and finally by the internal viscosity, which, however, becomes vanishingly small in the upper atmosphere. In fact, it is difficult to conceive of any serious resistance to motions in the higher strata, especially where convection is small or absent as in the stratosphere. Nevertheless, all these influences operate to retard or destroy motions. The great gyratory system of flow of cyclones, for example, all great and small motions of the air would soon run down and stop if not continuously maintained by an extraneous force or gradient. A part of the gradients which sustain motions is constantly expended in overcoming resistances and the final state of steady motion [6] is one in which the winds flow steadily *across* the isobars at a small angle. While this action resulting from resistance causes only a very small loss of *velocity*, 1 or 2 per cent, perhaps, below a state of frictionless flow, it will be accompanied by a very important and relatively considerable deflection of the winds amounting to 10° to 20° or more from the theoretical *direction.*

Motions near the Equator.—The deflective influence of the earth's rotation is zero at the Equator and very feeble for distances of several degrees of latitude either side thereof.[7] Within this wide equatorial belt motions of the air are of the simplest possible character, elsewhere the deflective influence is a more and more powerful disturbing factor attaining its maximum value at the poles. For present purposes it suffices to consider only horizontal pressure gradients and motions such as might occur over the oceans, because all winds are chiefly parallel to the ground, which is nearly horizontal in all but a few cases.

[6] The term "steady motion" with the same meaning as herein seems to have been introduced by Oberbeck, Mech. Earth's Atmos. Abbe. Smith. Miscl. Coll. 843, p. 177.

[7] The tropical hurricane has its infrequent and mysterious origin on the borders of the region here considered. This phenomenon, however, is an entity in itself, but it plays no part in the present considerations. The mathematical basis for the dreaded velocities of its wind systems is, however, completely understood and expressed by equation (15) for horizontal motions.

Imagine a small portion of free air, a cubic centimeter of air, for example (fig. 3) in a region where the pressure is high at the left and low at the right, as suggested by the isobars B_1 B_2 B_3, etc. Each face of the cube is subjected to a pressure over the whole surface which may be represented by the several forces $p_1, p_2, \ldots p_6$. In addition, the cube has mass and is pulled downward by gravity, represented by the relatively small force $W=\rho g$, in which $\rho=$ the mass of the cubic centimeter of air under the given conditions, and g is the acceleration of gravity. Now, since the pressure is assumed to diminish steadily from left to right, p_1 will be greater than p_2, and the cube will be urged toward the right by a force $\delta p=p_1-p_2$. Since we assume there is no change in pressure in the direction parallel to the isobars, then the two pressures p_5 and p_6 are equal and neutralize each other so far as motion of the cube is concerned, and may therefore be disregarded. Finally, p_4 must be greater than p_3 by just enough to make

$$p_4=p_3+\rho g \tag{8}$$

which is the equation of forces in the vertical. If p_4 is too great, the body *will be pushed vertically upward*, or

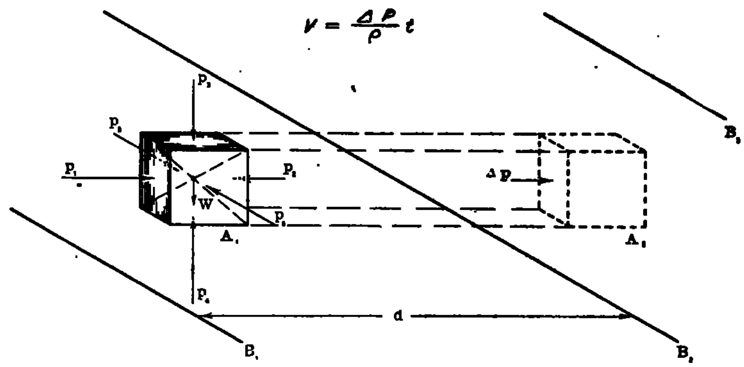

FIG. 3.—Forces resulting from atmospheric pressure and gravity upon a cube (1 c. c.) of air, weight W, and causing motion in the direction A_1 to A_2.

as it is commonly but less correctly stated, "it will ascend." If p_4 is not great enough, the body of air is said to be heavy, that is, it is insufficiently supported by the pressure of the surrounding medium, and it descends or falls under gravity.

We here assume the forces in the vertical are in equilibrium and may therefore be disregarded. Assuming further that our cube of air is not acted upon by any other forces than surface pressures, the excess force δp must move the cube down the gradient from a place of higher to a place of lower pressure, as from A_1 to A_2.

The force δp in this case is the pressure gradient and may be found from the spacing of the isobars by the following:

> RULE.—*The pressure gradient at a given place may be found from a weather map of the locality in question by dividing the difference in pressure between two isobars by their perpendicular distance apart.*

whence the equation:

$$\delta p=\frac{p_1-p_2}{1}=\frac{B_1-B_2}{d}K \tag{9}$$

in which K is a constant depending upon the units in which the isobars B and the distance between them, d, are expressed.

It will be instructive to calculate the velocity which a portion of air will acquire if free to move in a frictionless manner and acted upon by no other force than the pressure gradient δp.

Take a gradient of 4 millibars (4,000 dynes) per 100 kilometers (0.1 inch per 53 miles). While comparatively steep, nevertheless such a gradient is frequently shown on weather maps. The mass of a cubic centimeter of air varies, but may be taken at $\rho = .0012$ grams. From well-known equations of motion we have for velocity at end of t seconds

$$v = \frac{\delta p}{\rho} t \qquad (10)$$

or at the end of one hour

$$v = 26.8 \text{ miles per hour.} \qquad (11)$$

Hence air starting from rest and moving under the conditions assumed, gains velocity at the rate of 26.8 miles per hour. The distance traversed in the first hour would be 13.4 miles.[8]

Within the equatorial belt where the deflective influence is very small the motions of the air are almost strictly in accord with equation (10) and figure 3. Friction slows down the motion somewhat, but in all cases the air flows almost directly from the place of high to the place of low pressure. Owing to the absence of any causes adequate to produce and maintain marked contrasts, widespread uniformity of temperature and pressure is the ruling characteristic of this belt, accompanied by direct flow and easy intermixture of differing air masses, so that calms and light winds only are found, except in certain regions, as the Indian Ocean, where the monsoons prevail.

The surface winds, at least of the equatorial belt, are probably not true steady motions against balanced forces. This will be made more obvious in a later section treating of gradient winds for the globe.

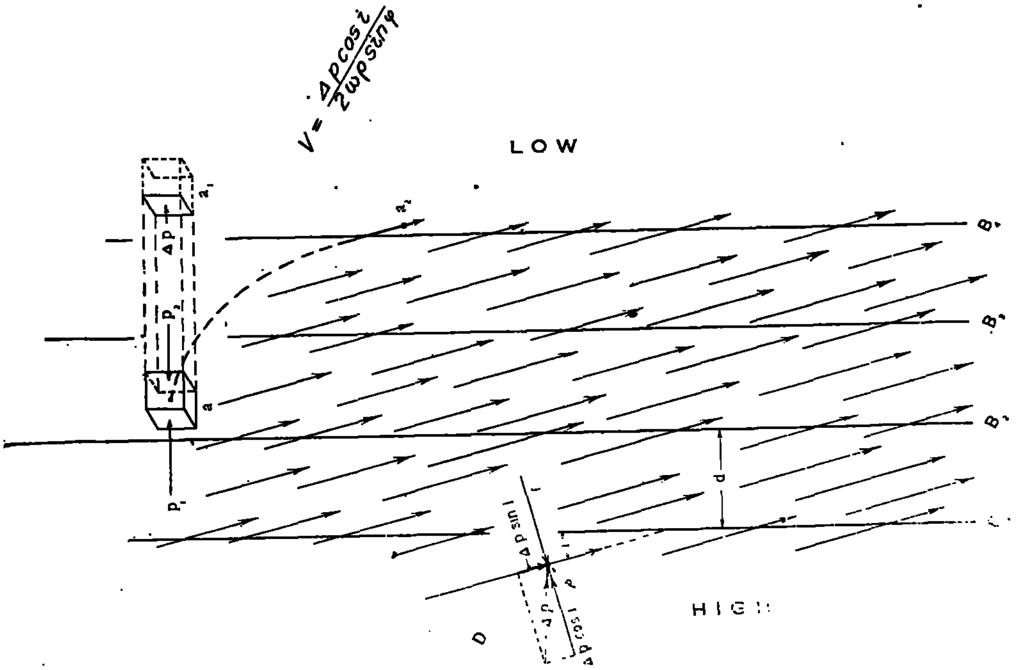

FIG. 4.—Diagram showing equilibrium motion of winds in straight lines under some frictional resistance for a pressure system represented by parallel straight-line isobars.

Motions controlled by deflective influence.—It was shown in figure 3 that a unit volume of air acted upon only by a pressure gradient δP would move simply down the gradient as from A_1 to A_2. (See also fig. 4.) We know, however, that no sooner is air set in motion than it is deflected to the right) or the left in the Southern Hemisphere) by the influence of the earth's rotation. Instead, therefore, of moving from a to a_1, figure 4, as one would be led to expect, the air will follow some such course as shown by the dotted line a to a_2. We can not trace the beginnings of atmospheric motions in any actual case, or follow the intermediate steps up to the attainment of steady uniform motion.[9] We can, however, clearly define the condition of steady motion such as shown in figure 4, which is intended to represent an extended system of straight and parallel isobars, B_1, B_2, etc., corresponding to high pressure to the left and low pressure to the right. Under such conditions the whole mass of air would flow in a steady stream along straight and parallel lines, such as indicated by the arrows, inclined at a slight angle across the isobars.[10]

The equation of this steady motion is easily deduced from the diagram of forces at D, figure 4, representing a unit volume of air. The pressure gradient δP is shown resolved into two components.

$\delta p \sin i$, acts parallel to the direction of motion and overcomes frictional resistance.

$\delta p \cos i$, acts normal to the motion and nullifies the earth's deflective influence.

That is,

$$\delta p \cos i = f = 2\, \omega\, \rho\, \mathrm{V} \sin \varphi$$

$$\mathrm{V} = \frac{\delta p \cos i}{2\, \omega\, \rho \sin \varphi}. \qquad (12)$$

Equation (12) gives the velocity of the "steady wind" for the gradient δp, the wind crosses the isobars at the angle i. If the friction diminishes, the parallel component, $\delta p \sin i$ will accelerate the speed, but f will also then increase and will further deflect the line of motion and diminish i, so that, clearly, when friction is wholly absent the angle i becomes zero; that is, the flow is strictly parallel to the isobars and we have the special equation for frictionless motion in a straight line (a great circle).

$$\delta p = 2\, \omega\, \rho\, \mathrm{V} \sin \varphi$$

$$\mathrm{V} = \frac{\delta p}{2\, \rho\, \omega \sin \varphi}. \qquad (13)$$

Equation (13) gives the velocity of the well-known gradient wind for straight isobars.

Steady winds and gradient winds.—The system of winds shown in figure 4 is intended to represent continuous

[8] Gold has given a much more extended treatment of this in Proc. Roy. Soc., vol. 80, May 25, 1908; also Mechanics Earth's Atmosphere, Abbe, Smith. Misc. Coll., vol. 51, No. 4.

[9] Shaw, Sir Napier: Manual of Meteorology, Pt. IV, p. 3, and Barometer Manual for use of Seamen. Metl. Off. 61, 1919, p. 8 et seq.

[10] Strictly speaking, such a system of parallel isobars must be a system of great circles having a common pole and therefore converging in both directions, but the amount of this convergence, within even an extended region such as here considered, is insignificant, and no appreciable error, therefore, is involved if we assume such isobars to be parallel.

motion at a uniform velocity under a uniform pressure gradient and a constant state of friction. It is entirely incidental that the isobars are straight instead of curved lines. There are two points to be emphasized: (*a*) that all the forces acting are completely balanced; (*b*) that one of these forces is friction, to overcome which the component of the gradient δp sin i must act directly along the path of the wind, which must be inclined across the isobars at the angle i. This example furnishes a complete case of steady winds under balanced forces. The active force of the pressure gradient δp is split up into two components; one, δp cos i, balances the passive action of the deflective influence; the other, δp sin i, overcomes the frictional restraints. The resulting *steady winds* blow *across* the isobars at an angle, i, which in the free air especially will be small.

A number of meteorologists are not careful enough in the use of terms and call any winds under balanced forces *gradient winds*. The writer strongly urges, in the interests of clearness of language, that the term *gradient wind* be reserved and used only for theroretical winds under balanced forces *when all friction is excluded*. By this usage, all natural winds when forces are balanced, are designated *steady winds* because friction is always present and the winds then blow across the isobars. True *gradient winds* flow strictly parallel to the isobars under balanced forces, friction being zero, a state never attainable in nature.

The steady winds of the globe.—The motions of figure 4 represent by far the greater portion of the permanent steady winds of the globe, because, especially over the ocean and away from the immediate proximity of cyclonic and anticyclonic centers, the isobars are long sweeping lines, the curvature of which is entirely secondary. These isobaric lines simply mark out the winding lanes and highways in the atmosphere along which the motions of the air must take place, never flowing quite parallel to the isobars but always cutting across the lines at some angle according to the amount of friction.

Such winds are perfectly free to flow either with or without friction over wide ranges of latitude at velocities moderate or otherwise depending quite entirely

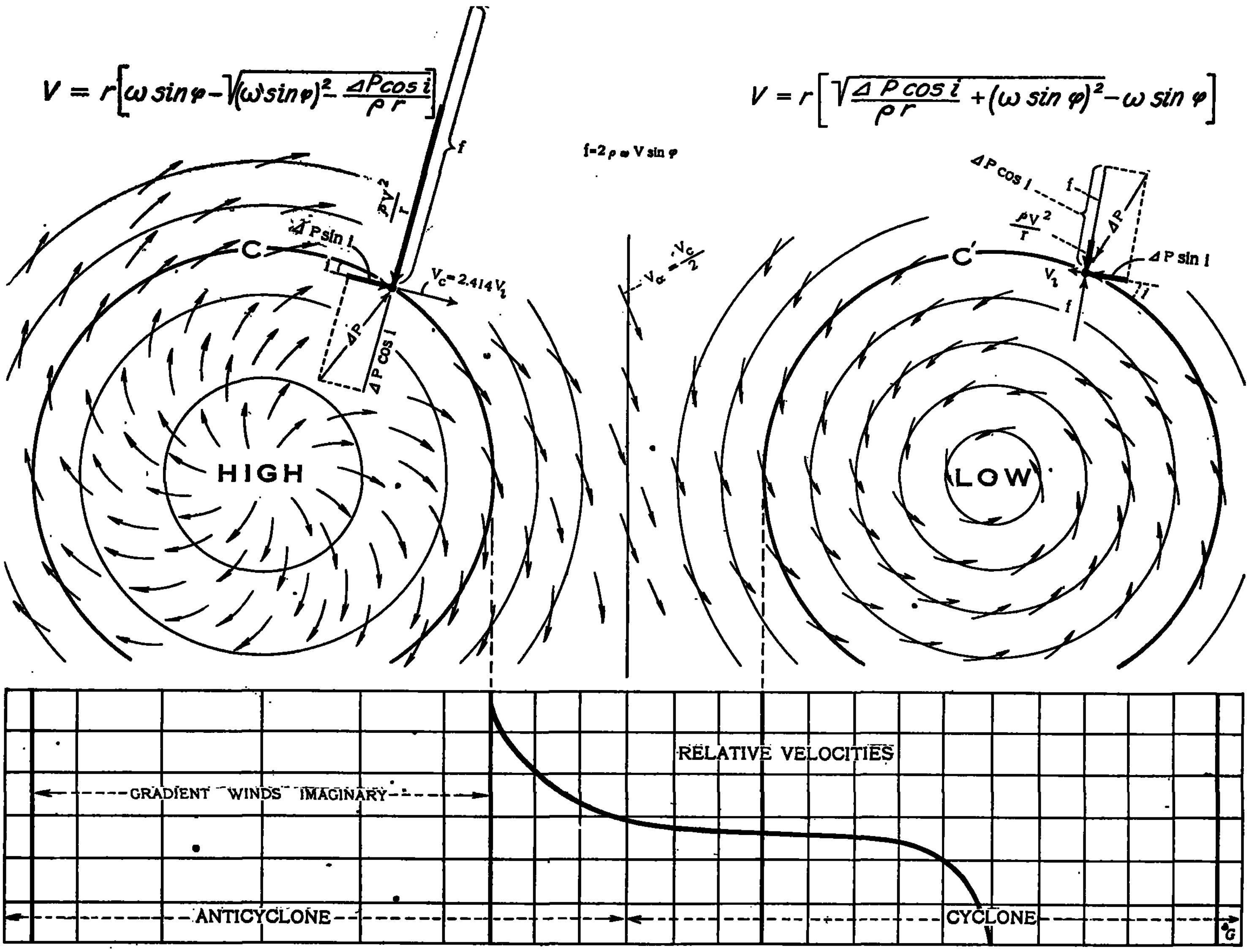

FIG. 5.—Idealized diagram to show steady motions of winds within and between highs and lows. For clearness winds are shown crossing the isobars at the appreciable angle i, due to friction. In the HIGH, C indicates the isobar with critical curvature. C' is an isobar of same curvature in the LOW. The active force causing all the motions is the pressure gradient Δp. Friction is overcome by the component of the gradient $\Delta p \sin i$. A force $\frac{\rho v^2}{r}$ is required to give the requisite curvature to the path of the winds; f is the deflective influence, which is balanced by the component of the gradient, $\Delta p \cos i$. All diagrams of forces are carefully to scale for a uniform gradient over the diagram. The lower portion shows the zone of imaginary gradient-winds in the High, and the relative velocities from the critical isobar of the high to the center of the low under uniform gradient.

upon the gradient. At every point the motion, or pressure, if motion is constrained, automatically satisfies the requirements of both the latitude and the slope effects, whatever the friction and the velocity.

Steady Winds with Curved Isobars.—Cyclones and anticyclones are well-defined cases of wind systems following strongly curved isobars and the fundamental conditions of steady motions are illustrated in figure 5. A "HIGH," anticyclone, and a "LOW," cyclone, are shown in juxtaposition and with closed isobars drawn as circles only to simplify the presentation. No suggestion is implied that this diagram represents Nature, except that when, in any actual cases, the gradients, curvature of the isobars, and friction are of the kind shown at any particular point on the diagram; then the velocities of steady motion will be of the nature indicated. Moreover, this motion will be that with reference to the existing pressure system *at the time and place.* It is well known that Highs and Lows travel at considerable velocities in definite directions. Therefore, the *actual* motion of the winds *over the ground* will be those compounded from (1) the motions appropriate to the system of curved isobars and (2) the motions of the system as a whole.

Any unit volume of air of mass ρ moving over a path of radius of curvature r, must be acted upon by a centripetal force $\frac{\rho V^2}{r}$ acting radially inward [11]. It is plain from the diagrams of forces in figure 5 that for winds inclined to the isobars at an angle i, the equations of motions become:

For LOWS:[12]

$$\frac{\rho V^2}{r} = \delta p \cos i - 2\omega\rho\ V \sin \varphi \tag{14}$$

from which

$$V = r\left[\sqrt{\frac{\delta p \cos i}{\rho r} + (\omega \sin \varphi)^2} - \omega \sin \varphi\right] \tag{15}$$

Similarly for HIGHS:

$$\frac{\rho V^2}{r} = 2\omega\rho V \sin \varphi - \delta p \cos i \tag{16}$$

which gives

$$V = r\left[\omega \sin \varphi - \sqrt{(\omega \sin \varphi)^2 - \frac{\delta p \cos i}{\rho r}}\right] \tag{17}$$

In all cases,

$$\text{Frictional resistance} = \delta p \sin i \tag{18}$$

For perfectly frictionless winds $\cos i = 1$ and equations (15) and (17) for strictly gradient winds become:

$$V = r\left[\sqrt{\frac{\delta p}{\rho r} + (\omega \sin \varphi)^2} - \omega \sin \varphi\right] \ldots\ldots \text{cylonic} \tag{15a}$$

$$V = r\left[\omega \sin \varphi - \sqrt{(\omega \sin \varphi)^2 - \frac{\delta p}{\rho r}}\right] \ldots\ldots \text{anticyclonic} \tag{17a}$$

When $r = \infty$ both become:

$$V = \frac{\delta p}{2\omega\rho \sin \varphi} \quad \text{(13) straight isobars.}$$

Figure 5 shows the directions of motions for the Northern Hemisphere, the directions being reversed for the Southern Hemisphere.

[11] The value of r when large should be taken with due regard for the curvature of the earth, but r can never be very accurately determined in practical cases and the consideration of the earth's curvature may generally be disregarded.

[12] Equation (15) may be written with the + sign in the second member for clockwise rotation in the cyclone. J. S. Dines has called attention to this interesting possibility. MONTHLY WEATHER REVIEW, Feb., 1919, 47: 87.

While equations (15) and (17) are basic and fundamental as defining the general steady motions of the atmosphere, *yet diversity of conditions, irregularity of pressure distribution, changes with time before a state of equilibrium is attained, errors in mapping supposed conditions, and other factors combine to make natural winds in particular cases differ widely and frequently from the theoretical deductions herein presented.* It is worth while, however, to develop somewhat fully certain important results based upon assumed ideal conditions, because these will aid greatly in reaching a full comprehension of atmospheric circulation.

A few of the things equations (15) and (17) tell us as to the possibilities of steady atmospheric motions follow:

(1) When the flow is frictionless $i=0$ and the equations are then the equations for gradient winds and the flow is strictly parallel to the isobars.

(2) When $r = \infty$ both equations reduce to (12) or (13) for straight line isobars and motions, with or without friction, as the case may be.

(3) When δp and friction $=0$, the equations determine the frictionless motion of a body set in motion at a velocity V.

(4) From equation (15) for lows it appears that notwithstanding diminution of air density as the pressure falls there seems to be no limit to the steepness or intensity to which the pressure gradient in *cyclonic* motion may attain. The velocity V may become indefinitely high as the radius of curvature r becomes smaller and smaller, provided the gradient is sustained. This corresponds to the enormous velocities of the winds found in the funnels of tornadoes and near the centers of tropical cyclones.

(5) In contrast with the conditions shown for LOWS, equation (17) for HIGHS leads to very different results. Gold [13] has pointed out that the value of V from (17) will be imaginary for all values of $\frac{\delta p \cos i}{\rho r}$, which are larger than $\omega^2 \sin^2 \varphi$. It is also obvious that when $\frac{\delta p \cos i}{\rho r} = \omega^2 \sin^2 \varphi$ the velocity V will have its maximum value which can be shown to be

$$V_c = \frac{\delta p \cos i}{\omega\rho \sin \varphi} \tag{19}$$

The corresponding radius of curvature of the wind path (of the isobar, if $i=0$) will be

$$R_c = \frac{\delta p \cos i}{\rho\omega^2 \sin^2 \varphi} \tag{20}$$

From these equations we learn that for every anticyclone the condition of the steady flow of winds nearly parallel to the isobars can not be satisfied within the central regions of the high because the radius of curvature is too short and very high velocities are necessary.

Critical isobar.—The word *critical* has come to be applied to the limiting conditions in anticyclones under which "gradient" or "steady" winds become possible. Thus we have "critical isobar," "curvature," "velocity," and the like.

The *critical isobar* of a HIGH is significant in atmospheric motions in that, within the area inclosed by it, the winds blow outwardly at a relatively high angle across the inner isobars. Unless friction is too great, the speed of

[13] Mechanics of the Earth's Atmosphere, Abbe, C. Paper by Gold, E., Smith. Misl. Coll., No. 4, p. 113.

outflow will be accelerated by the gradient and attain a relatively high velocity. The flow then comes to be more and more deflected by the earth's influence until it reaches the isobar of critical curvature, where it comes into a state of steady motion at its maximum, or the *critical velocity*.

Because of these conditions we find that beginning with small gradients and gentle winds at the central portions of highs we may look for winds of the highest velocities far away from the center and along the *critical isobar*, whose diameter is generally considerable. Beyond the isobar, especially in the direction of an adjacent low pressure system, the curvature of the isobar and also the speed of the wind diminishes as the isobars merge into parallel straight lines. Passing onward from this region toward the center of a LOW the velocity decreases as the curvature of the isobars increase, but generally in nature the gradient in lows increases toward the center, and at times becomes very steep with high gradient wind velocities.

Comparing equations (12) and (19) we see the steady wind along the isobar of critical curvature is exactly twice as great as the steady wind for parallel straight isobars, and for the same pressure gradient and angle i.

While the steepest gradients and the highest winds occur with cyclonic conditions, nevertheless gradient winds of the anticyclone have much higher velocities than those of the cyclone, when the pressure gradient and curvature of isobars are the same in both cases. It can easily be shown that regardless of the latitude the velocity along an isobar of critical curvature of the anticyclone is $\frac{1}{\sqrt{2}-1}=2.414$ times the wind along an isobar of the same curvature and same pressure gradient in a cyclone.

In the foregoing the configuration of the anticyclonic system has been regarded as closely circular in outline. In nature, however, HIGHS repeatedly occur of gentle gradients and enormous extent, not only laterally, but especially extended longitudinally so as to constitute a veritable ridge of high pressure, overspreading an extended region. Such are frequently found over ocean areas. The isobaric lines of such a system lack curvature and are practically parallel straight lines in the direction along the axis of the HIGH, and the outflow therefrom, when the equilibrium stage has been attained, conforms closely to the motion shown in figure 3. The parallel straight lines are in fact the isobars of *critical curvature* for this condition and the *critical speed* in this case, equation (12), as we have seen, will be only half as great as the maximum possible speed for the same gradient in a HIGH with circular configuration; that is, with a radially divergent gradient.

These considerations explain why it is that weather maps frequently show great areas of high pressure in which the gradients are very gentle for a long distance from the center, beyond which critical distance the isobars, often nearly or quite straight lines, are crowded close together, marking a strong gradient and accompanied by high winds.

All this means simply that the air flowing initially *down the gradient* while gaining velocity finally attains its equilibrium state in steady motion nearly *parallel to the isobars*. The air flows across the isobars with difficulty and therefore the transfer of great masses of air from one place to another in order to satisfy a pressure deficit can not be easily and quickly effected. The principle of frictionless gradient winds means flow *perpendicular* to, not down the gradient. This thought leads to an interesting dynamic paradox, which may be stated thus:

Dynamic paradox.—If air is caused to flow from one place on a rotating globe to a more or less distant place (not near the equator) by a fixed pressure gradient steadily maintained, then paradoxically the greater the friction the more direct and easy the flow; the less the friction the more difficult the flow. If the friction is zero the flow to the distant place may be impossible; but the maximum velocity in the steady state will be finite, depending upon the gradient. (See discussion of the critical isobar in the HIGH.)

The permanent low pressure at the poles may be explained by noting that the outflow of air from this region as a whole is over the surface and in the lower strata, where friction, convection, etc., are greater than in the case of incoming air at higher levels. Hence, the region experiences a deficit of air until a pressure gradient is built up under which the inflow and outflow are just balanced. This principle is of course obvious enough but its wide application in connection with the existing permanent contrasts of pressure needs to be emphasized and recognized.

The following table presents instructive data giving gradient winds computed from equations just discussed for different latitudes:

TABLE 2.—*Gradient winds and curvature of isobars for critical conditions in different latitudes.*

[Isobars spaced 100 miles per $\frac{1}{10}$ inch. Density $\rho=.0010$ grams per cc. corresponds to free air at about 1 mile. Units are miles and miles per hour.[14]]

Latitude (degrees).	Critical velocity (High).	Radius of critical isobar (miles.)	Velocity same curvature in Low.	Velocity for straight isobars.
0	Infinite.	Infinite.	Infinite.	Infinite.
10	372	8,160	154	186
20	189	2,100	78.2	94.4
30	129	984	53.5	64.5
40	100	595	41.6	50.2
50	84.3	419	34 9	42.1
60	74.5	328	30.9	37.3
70	68.7	278	28.5	34.3
80	65.5	254	27.1	32.8
90	64.6	246	26.8	32.3

[14] A more complete table in metric and English units for various gradients has been published by the writer in Smithsonian Meteorological Tables, fourth edition, 1918, Tables 42 and 43.

V. EARTH'S SURFACE NEARLY GEOIDAL AND FRICTIONLESS.

Since our present object is to consider certain fundamental influences governing the great general motions of the atmosphere, we may regard the irregularities in the geoidal surface represented by the waves of the sea, for instance, or even the elevations of the islands and continents, however rough their topography or lofty their mountains, as inconsequential or of secondary influence. It is well known these topographic reliefs are insignificant as roughness on even a large-sized globe. On the other hand, it is also true that the whole atmosphere, although without definite outer boundary, is itself only a relatively thin layer. For example, the troposphere within which all the great convective actions occur embraces fully 75 per cent of the atmosphere yet has a thickness of only 7 or 8 miles. On a 12-inch globe a layer of the thickness of an ordinary blotting paper might represent this air, and the most lofty mountains almost pierce it at several points, and very extended mountain ranges project into it to heights of 2 or 3 miles. The actual superficial area covered by these real obstacles to the great motions of the atmosphere, however, is but a small fraction of the whole surface of the globe, of which perhaps more than 90 per cent may, for present purposes, be regarded as oceans and smooth lowlands. Accordingly, in following the statements presented herein the reader is requested to keep vividly in mind the actual smoothness of the real geoidal surface of the earth and for the time being forget the irregularities which appear so exaggerated to our limited perceptions.

Careful reflection upon the foregoing seems to justify the statement that the geoidal form of the earth in the lower strata of the atmosphere, as a matter of fact, is realized with great perfection because the highly mobile air itself, rather than even water, constitutes the real surface. Portions of the air which remain at rest or nearly so because of terrestrial roughness themselves constitute the geoidal surfaces upon which the great motions of the air take place in a nearly frictionless manner. We are not required to think of motions over rough, irregular land areas, but simply of the free masses of moving air gliding easily and almost without friction over relatively thin layers of air held partly stationary by, and filling up, the actual roughness of the globe, in a manner which transforms, or literally lubricates, the actual solid figure into one of great smoothness over which the atmosphere moves in a practically frictionless manner. Obviously, however, turbulence and the convection of large masses of air offer very important obstructions to the movements in question and add to the smaller losses of energy due to the friction at the earth's surface.

These considerations lead indeed to the view that the actual motions of the atmosphere as we know them are in reality very nearly frictionless motions, and if every bit of frictional resistance were removed the motions under existing gradients would be much the same as at present. The energy now dissipated by friction would be saved and wind velocities would be slightly increased. Friction now permits and facilitates a very material interzonal flow, which would be suspended without friction, in which case, after a time, the present contrasts of temperature between the equator and the poles would be changed, causing changes in the pressure gradients. There is nothing, however, attendant upon the removal of all friction to support Ferrel's conception of the frictionless circumpolar cyclone.

Throughout all that precedes an effort has been made to acquaint the reader fully with a correct view and understanding of the forces, conditions, and influences causing and modifying the motions of the atmosphere and of bodies on a rotating globe. With such a correct understanding in mind the serious errors in the existing literature of the subject can be most clearly presented and discussed in connection with quotations such as follow:

VI. Citations From Authorities Showing Fallacies Herein Discussed.

There are two essentially different aspects of what is actually one fundamental fallacy leading to inconceivable wind velocities in atmospheric motions. (1) The equations offered by the mathematicians such as Ferrel, Helmholz, Oberbeck, and Bigelow, for example, all lead to very high velocities, which are difficult to explain. (2) Another class of writers, such as Davis, Hann, Angot, Milham, McAdie, Humphreys, and others, have endeavored to present the principles of atmospheric motions in popular language, but have introduced misinterpretations of their own so as to increase the confusion and misrepresentations. Since a fundamental difficulty still remains after clearing up the fallacy in the writings of the popular writers, it seems best to begin with the latter and then discuss the faults in the work of the mathematicians.

CITATIONS FROM POPULAR WRITERS.

DAVIS, W. M.—Mention has been made already that Davis first called attention to the friction fallacy when discussing Hadley's faulty theory concerning the change of velocity of winds with change of latitude.

Although on the right track in these matters, he was not consistent and later fell into the fallacy of super-hurricane velocities growing out of vortical motions. After having applied the law of the *conservation of areas* to the vortical motions of water in a basin discharging through the center and likened the same to the vortical circulation of the atmosphere around the poles, Davis says (Elementary Meteorology, p. 110):

> 136. *Cause of low pressure around the poles.*—If the explanation of section 134 be now applied to the atmosphere, with the supposition that there is no loss of velocity by friction or other resistance, it is clear that an excessive velocity and a still more excessive centrifugal force would be developed in the circumpolar vortices. It should be noticed that the eastward motion of 1,000 miles an hour that the air has over the equator is increased as the overflow approcahes the pole; at latitude 60°, where the distance from the axis is half what it was at the equator, the eastward velocity has doubled; that is, it has become 2,000 miles an hour, or 1,500 miles faster eastward than the earth's surface at that latitude. Forty miles from the pole it would be 100,000 miles an hour; and so tremendous a velocity on so short a radius would suffice to hold the air away from a closer approach to the pole, if it could, indeed, approach so close as this; at any less distance there would be a vacuum.
>
> But the action of friction and other resistances can not be neglected. The presence of almost as great an atmospheric pressure in the polar regions as at the equator assures us that the imaginary case of no friction is far from the actual case. Although the resistances suffered by the upper air currents can not be great, they successfully prevent the realization of the enormous circumpolar velocities that would result in the case of no friction and no intermingling of currents.

The correctness of these statements may be questioned from two points of view.

(1) Friction and the dissipation of energy by convection, turbulence, eddy motions, viscosity, and all imaginable internal wastes are of the greatest importance and can not be neglected or overlooked in the full analysis of atmospheric motions. On the other hand, it is wrong to imply that slight atmospheric friction and other resistances suffice to prevent the attainment of the inconceivable eastward velocities of 1,500 miles per hour at latitude 60° and 100,000 miles per hour at 40 miles from the poles, which the computations by the law of equal areas give. It is very clear there is a serious error here. Friction plays an important part in the circulation of the atmosphere, but it does not play the part ascribed to it in the statement quoted.

(2) The law of equal areas applies to motions *under a central force*. Now in the case of the vortical whirl of water, or even of the cyclone, the hurricane, or the tornado, the only *active* central force is the convergent pressure gradient in the system. Gravity is *not* in such cases an active central force causing the gyrations except to the extent its action is expressed as a pressure gradient. Again, in the case of the circumpolar cyclone, practically all the authorities treat it just as if gravity were the active central force. This is entirely erroneous. The only circumpolar winds which are possible under any assumptions are simply the winds which would occur anywhere with the same pressure gradient, friction, and deflective force. We have already shown that the law of equal areas in connection with rotation about the

earth's axis can not act alone in any of these cases. It must act with the slope effect, simply as a passive influence which only guides the moving masses in a manner such that the law of momentum is satisfied at every point *with no change in velocity* unless such is required by actual changes in the gradient or the friction.

On a rotating earth the whole effect of rotation on atmospheric motions may be summed up in these words:

Rotation compels motions to become gyratory and the gyrations must always be in particular directions.

The deflective influence, *f*, varies in value from zero at the equator to a maximum value at the poles. This force represents the whole influence of the rotation. It is proportional to, but is powerless to produce or change velocity. On a stationary earth the motions of the atmosphere would not in general assume a gyratory character. Nevertheless, a gyratory motion could be set up just as it is easy to set up vortical outflow of liquid from a basin. In such cases the direction of gyration depends entirely upon the initiating cause and might be in either direction *ad libitum*, whereas rotation of a globe compels relative motions thereon to become gyratory and in a particular direction. But the motions must be set up first by some extraneous force; a pressure gradient as a rule. The advocates of polar hurricanes do not show the source of the inconceivable forces or gradients which alone could produce the excessive velocities. Air at full pressure flowing directly into a vacuum—a maximum conceivable gradient—could not attain the excessive velocities claimed for frictionless polar circulation. As is fully shown in Section IV by the equations for steady winds either with or without friction, acceleration of velocities ceases immediately when the pressure gradient and the deflective force are balanced. This absolutely fixes the possible velocity in any case and these are always moderate.

To move a body northward or in any direction on a frictionless stationary globe, we need only to push it northward or in any direction in which motion is desired. To move it northward on a globe rotating from west to east *we must push the body, not northward, but constantly westward*, when it will move westward to a slight extent but northward indefinitely unless started exactly at the equator. On a stationary globe steady motions take place in the line of action of the producing force. On a rotating globe, *perpendicular* to said line of action if frictionless, and nearly perpendicular if friction is slight. These truths have long been known, but writers have neglected their consistent application.

It will be shown later that anticyclonic gyrations are impossible on a stationary earth.

MILHAM.—The fallacy by this author is found in the following passage:[15]

146. The effect of this deviation to the right on the air masses which, due to convection, are moving from the equator toward the poles on the outside of the atmosphere must now be considered. Instead of moving directly from the equator poleward, the air masses will be deviated to the right in the Northern Hemisphere and become more and more a west air current encircling the pole in a great whirl. It is a principle of mechanics that whenever a rotating body is not acted upon by outside forces, the moment of momentum must remain a constant. The formula for the moment of momentum is ΣMVR, where M represents the mass of each particle of the rotating body, V the velocity of the particle, and R the radius, that is, the distance of the particle from the center of rotation. This product MVR must be summed up for all the particles of the rotating body, and thus ΣMVR represents the moment of momentum of the body. As this ring of whirling air about the pole approaches it, the mass remains constant, the radius is decreasing, and thus the velocity must steadily increase, and it has been computed that, if the velocities were not held down by friction, they would amount to hundreds of thousands of miles per hour. A whirl of air with these high velocities must cause centrifugal force, and this centrifugal force will hold air away from the pole, thus causing a diminution in the barometric pressure. The amount of land at the north pole is much greater than at the south pole. One would thus expect wind velocities and the diminution in pressure to be larger at the south pole than at the north pole.

As in the case of Davis, no distinction is made by Milham between the earth's deflective influence and the operation of the law of equal areas. This is the more remarkable because earlier, even on the same page, this author says: "The earth's rotation influences air moving in any direction and the velocity does not increase as the equator is approached." Why, then, should it increase as the pole is approached?

VON HELMHOLTZ [16] encountered the excessive velocities required by the law of equal areas and overlooked the correlative influence we herein designate the law of the geoidal slope

On page 82 we find:

For air that is resting quietly at the Equator in the zone of calms and is thence pushed up to the latitude of 10°, this expression gives the acquired wind velocity 14.18 meters per second, and similarly for air pushed up to latitude 20°, 57.63 meters, and for 30°, 133.65 meters per second.

Since 20 meters per second is the velocity of a railroad express train, therefore these numbers show without further consideration that such gales do not exist over any broad zone of the earth. We therefore ought not to make the assumption that the air which has risen at the Equator reaches the earth's surface again unchecked in its motion even 20° farther northward.

The matter is not much better if we assume the atmospheric ring resting at some intermediate latitude. In that case it would give an east wind at the Equator, but a west wind at 30° latitude; but both velocities would far exceed the ordinary velocities of the observed winds.

Since now in fact observations do demonstrate a circulation of the air in the trade-wind zones, therefore the question recurs: By what means is the west-east velocity of this mass of air checked and altered? The resolution of this question is the object of the following remarks.

Helmholtz then sets up equations from which he determines the conditions of equilibrium between superincumbent strata with differing motions and temperature. In arriving at these equations he introduces terms representing the effects of the *centrifugal reactions* which we call the slope effect and then considers the turbulent and whirling intermixture which result from instability of strata. There seems to be no logical justification in classifying the slope effect with friction, turbulence, etc., and it seems quite uncertain what the effects of turbulence, etc., would be in his equations if he had treated the slope effect in a proper manner. Helmholtz's conclusions read:

From these considerations, I draw the conclusion that the principal obstacle to the circulation of our atmosphere, which prevents the development of far more violent winds than are actually experienced, is to be found not so much in the friction on the earth's surface as in the mixing of differently moving strata of air by means of whirls that originate in the unrolling of surfaces of discontinuity. In the interior of such whirls the strata of air, originally separate, are wound in continually more numerous, and therefore also thinner layers spirally about each other, and therefore by means of the enormously extended surfaces of contact there thus becomes possible a more rapid interchange of temperature and equalization of their movement by friction.

HUMPHREYS, W. J.—Statements claiming that atmospheric frictions, turbulence, etc., prevent the excessive velocities called for by the law of equal areas appear anew in the publication entitled "Physics of the Air," Journal of the Franklin Institute, November, 1917, page 660 (Book Form, p. 131), as follows:

Hence the velocity of the transferred air [from lat. 30° to lat. 60°], in question with reference to the surface is $v'-s$=1,036 miles per hour=436 meters per second.

[15] Milham: Meteorology, par. 146, p. 161, edition 1912.

[16] Mechanics of the Earth's Atmosphere. Translations by Cleveland Abbe. Smithsonian Miscl. Col., 843, 1891, V. On atmospheric motions.

As a matter of fact, no such enormous velocities of the wind as the principle of the conservation of areas would lead one to expect in the higher latitudes are ever found, either at the surface or at other levels. This, however, does not argue against the applicability of the principle itself, but only shows that in the case of atmospheric circulation there are very effective damping or retarding influences in operation.

The resistance due to the viscosity of the atmosphere is one of these retarding influences, but its effect probably is very small. A larger effect doubtless comes from surface turbulence induced by trees, hills, and other irregularities. A still greater velocity control, probably so great that all others are nearly negligible in comparison, except near the surface, is vertical convection. This phenomenon leads to extensive interchanges between lower and upper layers of the atmosphere, thus indirectly increasing the effect of surface friction probably several fold and tending to bring all the lower, vigorously convective, portion of the atmosphere to a common velocity. Because of these several means of control the actual wind velocity everywhere is different and at high latitudes much less than it otherwise would be.

Not only is the velocity of the wind changed through change of latitude, but also the rate at which its direction with reference to the surface of the earth varies or tends to vary.

It seems obvious, on sober reflection, that the slight frictional resistances itemized in the foregoing could not possibly reduce the inconceivable velocity of 1,036 miles per hour to even hurricane winds, much less hold it down to the average gentle winds of our everyday experience.

Like others, this author disregards the operation of the slope effect. In fact, in the publication cited all discussion of the slope effect, the *dual* nature of the deflective influence of the earth's rotation and the interrelation between these concepts and the law of equal areas is entirely omitted. A method is employed for demonstrating the deflective influence which is of doubtful value because it conceals from the pupil the very information it is most important for him to know and fully understand, namely, that, as Ferrel so clearly showed at the beginning, the passive deflective influence is the resultant of two wholly separate and independent inertia reactions. These are the momentum effect in the one case, and a lateral component of the so-called centrifugal effect or the slope effect in the other case.

Attention is called also to the the further misrepresentations in the closing paragraph of the quotation from Physics of the Air. The opening words imply the velocities of winds change with change of latitudes, whereas it has been shown repeatedly that change of latitude does not in itself lead to changes in the velocity of winds and freely moving bodies.

Furthermore, the rate of change of wind direction with reference to the surface of the earth, due to the deflecting influence of the earth's rotation, is not correctly expounded in the text following the quotation. The numerical values of angular changes per hour given in the table on page 665 (136 Book edition) are not changes in the direction of the wind, as the context clearly implies, but are simply the angular turning of the ground, whereas the changing in the direction of motion of winds and freely moving bodies is double the amounts given, as already stated in Section III of this article.

MARVIN, C. F.—The writer acknowledges having been formerly misguided by the concurrent writings of many high authorities on these questions and himself subscribed to the friction fallacy. (See Weather Forecasting in the United States, p. 17.)

These citations, and several others of like import which might be added, are erroneous, because the writers have disregarded the slope effect and have ascribed to friction the part it plays in controlling the operation of the law of equal areas.

The soundness of any of the statements of excessive velocities quoted can not be justified on the ground that they are correct for the conditions upon which they rest. This defense is unacceptable. The statements quoted were written for the instruction of uninformed readers upon a very obscure question of dynamics, and the statements should be correct on their face. The reader can not be expected to supply missing information. As a rule, the absence of friction is the only condition specified and the inference is permitted and generally entertained that bodies once set moving in latitude without friction would continue to change their latitude indefinitely. In all the statements dealing with excessive velocities under the law of equal areas the reader is left ignorant of the peculiar conditions under which the statements might be correct. For example, indefinite changes of latitude with excessive velocities in longitude would occur on the assumption, which however is nowhere stated, is not tenable and is not even generally recognized as a condition—namely, that the pull of gravity *on the moving body* is exactly perpendicular to the smooth geoid. Such a condition is wholly impossible in nature.

Again, waiving the condition just stated, reflection will show that the excessive velocities claimed in any case, as for example, the velocity of 1,036 miles per hour, might be attained by a body reaching a latitude of 60° *provided it was started northward at 30° with a velocity of 1,036 miles per hour.*

We may feel confident that these limiting conditions would have been fully mentioned if clearly perceived by any of the writers, since to omit such important factors consciously would have been inexcusable. On the other hand the complete absence of any allusions to such limitations justifies the claim that the statements are erroneous as they stand. In any case, moreover, it is the operation of the neglected slope effect, not friction, that prevents the excessive velocities.

What we consider the fundamental fault still remains, and for this chiefly the mathematicians are responsible. I think we may fairly say that the original error starts with Ferrel and has been perpetuated for the last 60 years in all standard writings.

Ferrel's writings are very clear on the subject of the deflective influence, and its two components, both of which were properly included by appropriate terms in his general equations.

Even to the present day it is difficult to overestimate the remarkable originality of Ferrel's work especially when we consider how little was known of general meteorology at the time, 1858, he produced, in almost its finished state, his mathematical theory of the circulation of the atmosphere.

In spite of this, however, he appears to have fallen into important errors arising from a disregard of the control exercised by the slope effect which led to serious misconceptions of inconceivable velocities to which his final equations lead when friction is assumed zero or negligible.

Citations from mathematical papers.

It is believed the crucial question at issue may be stated to be—

What is the nature of the *frictionless* circulation of the air of a polar hemisphere assumed to be warm at the equator and cold at the poles?

For such studies the two polar hemispheres of the earth may be regarded as entirely independent units, completely separated by the plane of the equator extended, across which all reactions are neutral.

Ferrel approached the solution of this question by first assuming the air at rest and no temperature gradient, the air being given an initial impulse in the plane of the meridians. The result gave him his circumpolar cyclone with polar winds of inconceivable velocities whereby the air from its own motion would entirely recede from the poles, be greatly depressed at the equator and form a bulge of high pressure at lat. 35° 16′ [17].

On page 195 of Recent Advances, he says:

> Hence in this case of no friction between the atmosphere and the earth's surface any stratum of equal pressure, however rare it may be and however high it may be in the equatorial and middle latitudes, must be brought down to the earth's surface near the poles.

This concept necessarily entails a vacuous region of material extent "near the poles," and is so shown in figure 6. This condition of pressure distribution involves motion, and of this Ferrel says on the next page:

> In the preceding results the motions are due to an initial impulse giving the initial velocities u_0, v_0 and x_0 and not to any constant temperature disturbance, since α, which is the only quantity depending upon temperature, has been treated as a constant.

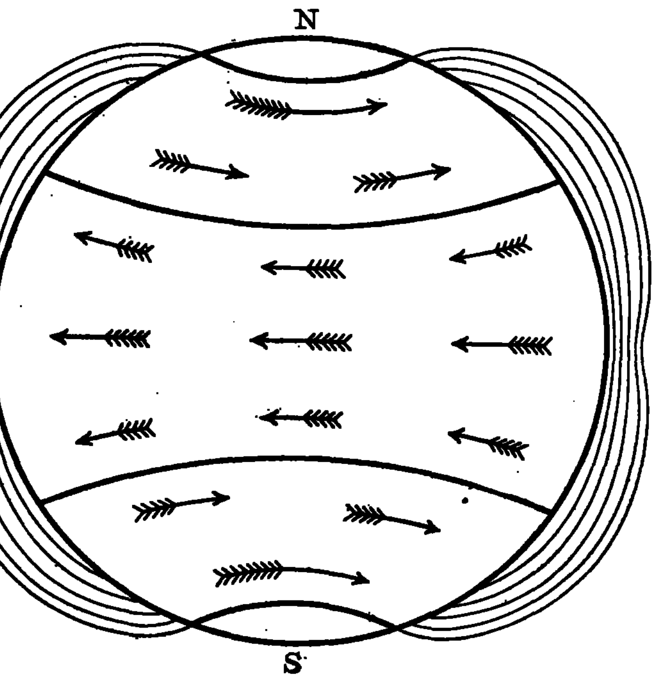

FIG. 6.—Reproduced from Ferrel, showing his frictionless circumpolar cyclone.

This concept has been practically universally accepted by meteorologists and taught up to the present time, subject, of course, to the assumptions upon which it is based.

The superhurricane winds at different latitudes required by this theory have been given already in the last column of Table 1 and are repeated in Table 2 below.

These conclusions of Ferrel very largely underlie all his subsequent theoretical deductions in which he endeavors to define the general circulation with temperature gradients and friction properly included. To the last he seems to have been able to satisfy his own mind at least that friction would suffice to check the incredible velocities demanded by this theory at both the equator and the poles.

Sober contemplation of such a circumpolar hurricane even granting frictionless motion simply compels the conclusion that it is obviously erroneous, unreasonable, and impossible. If such is the case can we still believe his more complete solution of the complex problem with temperature gradients and friction included is nevertheless valid? Probably not.

As in the case of the popular writers the soundness of the conclusions reached by the mathematicians can not be justified on the ground that they satisfy assumptions. Let us examine this question briefly.

While for directness of presentation we shall confine our analysis to Ferrel's writings as the leader in these matters, nevertheless, his mistakes must be shared by all those who have followed in his footsteps.

He says of his frictionless circumpolar cyclone (see citation above): "In the preceding results the motions are due to an initial impulse giving the initial velocities u_0, v_0, and x_0". Elsewhere he makes it clear he assumes the initial state of the air before receiving the impulse to be that of "rest relative to the earth's surface." The assumption relative to the initial impulse is vague and indefinite. It is difficult to imagine any kind of an initial impulse that would satisfy the idea in Ferrel's mind, or even one that would produce results such as he claims would follow. We may assume each particle receives an impulse peculiar to itself, but clearly such a condition involves great complexity, and it is certain Ferrel never would have omitted to cover such an important point if such a thought were entertained by him. On the contrary, we may be sure Ferrel thought that, in the absence of friction, particles set moving northward would continue moving northward indefinitely. For example, imagine that a great ring of free upper equatorial air is given a powerful impulse projecting all particles exactly poleward. Helmholtz entertained a view somewhat like this. Would such an impulse induce a circumpolar cyclone like Ferrel's? Not at all. Every particle disturbed by the impulse, if close to the equator, would tend to execute motions such as Whipple has demonstrated, and if more distant from the equator the motions set up would tend to circular paths, according to the explanation of figure 2. If we wholly disregard the action of the slope effect, as we feel confident Ferrel and all those who have accepted his view unwittingly did, it seems quite certain that the assumed atmospheric ring once started northward would continue to cross lines of latitude indefinitely, and at the same time take on the high easterly velocities given in column 4 of Table 1. Assuming, further, that the northward motion of the ring is communicated without loss of energy, but reduction in northward velocity, to all the other particles of the polar hemisphere, it is easy to concede on an untenable basis, which neglects the slope effect, that a frictionless circulation like Ferrel's, with inappreciable motion in latitude but excessive eastward drift, especially in high latitudes, might be induced.

On this basis, however, Ferrel's circumpolar cyclone is erroneous because based upon an impossible condition which he and his followers have unconsciously adopted.

The writer is further inclined to believe that Ferrel's theoretical high pressure belt at latitude 35° 16′ is also fallacious and in the nature of a mathematical fiction, or the result of unnecessary or inapplicable assumptions.

However, this matter can not be lightly dismissed because Ferrel's entirely original work is strongly supported by like conclusions reached later by other mathematicians, such as Helmholtz, as already cited, Oberbeck, Bigelow, and others. This is made clear most easily by citations from Bigelow [18], who made a detailed com-

[17] Ferrel: Professional Papers, Signal Service, No. 8, p. 21, fig. 4; Recent Advances, Annual Report, C. S. O., 1885, pt. 2, p. 195, fig. 3; Popular Treatise on Winds, p. 118, sec. 81; also pp. 48, 67–69.

[18] Report on international cloud observations. Annual Report Chief of Weather Bureau, 1898–99, pt. 2, p. 588.

parative study of the mathematical work of Ferrel, Sprung, Guldberg, and Mohn, Oberbeck and Pockles. Bigelow was not fully convinced of the soundness of the conclusions reached by the mathematicians and considered the friction concept a perplexity. Taking up Oberbeck's work, which carried forward and improved upon the work of Guldberg and Mohn, Bigelow writes (p. 606):

Oberbeck's solution taken in connection with Ferrel's, constitutes the theory commonly taught by meteorologists. That it is partially correct, even admitting the limitations with which the solution was executed, is apparent, as it is verified by the general features of the circulation of the atmosphere. But there are, nevertheless, some important modifications which must be incorporated into this theory before it reaches a satisfactory statement. The first concerns the *return current* from the tropics toward the poles; the second the process of checking the excessive esatward *drift* in the higher latitudes; and the third the evaluation of the friction coefficient.

Bigelow by no means succeeds in clearing away the perplexity of "the process of checking the excessive eastward drift," although he devotes several pages of his cloud report to its discussion. He says (p. 607):

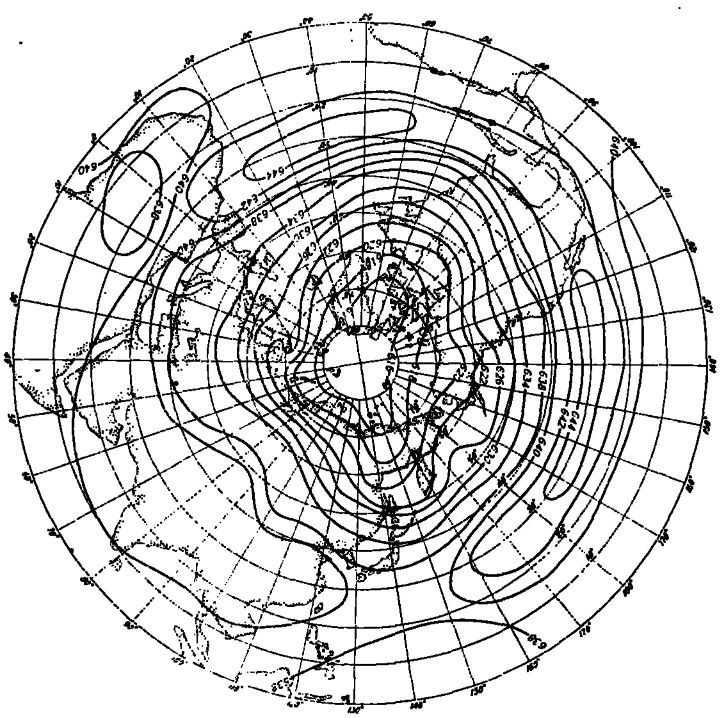

FIG. 7.—Isobars at 1,500 meters, January.

If the source of retardation conceived by Prof. Ferrel is not correct, is it possible to give another statement agreeing with the facts, and yet efficient in its operation to produce the required results? For it must be admitted that a very effective agency is in operation to act as a brake or check of sufficient power to reduce the theoretical eastward drift from the velocities of Table 122 [see Table 3] to those of Table 124. (Cloud report, p. 606.)

On page 614 he adds:

Fuller descriptions of these equations may be found in Prof. Ferrel's works. The chief impression regarding them is that they are detached in the discussion, one feature at a time alone being considered. Oberbeck's analysis has the advantage of much greater coherence and completeness, and in that respect is more satisfactory. At the same time it must be admitted that both systems give about the same picture of the general cyclone, if the assumptions introduced into the discussion are taken to be correct. Our criticism shows that these must be modified before the observed motions of the atmosphere are completely accounted for. It still remains, however, a very hard problem to solve practically.

In his later work, Atmospheric Circulation and Radiation, it seems Bigelow is still baffled and perplexed by the problem of friction and excessive velocities. On page 179, about friction [19] he writes:

This is a subject that has not been satisfactorily cleared up and it will require much careful research.

After deriving again essentially Ferrel's equations, Bigelow says (p. 191):

Ferrel discusses these equations and gives some approximately correct views regarding the general circulation. Oberbeck's treatment embraces the three equations of motion and the solution approaches more closely to the flow of currents actually observed. The complete intergration of the system is, however, more complex than has been admitted and the problem awaits a better treatment. The actual velocities and direction of motion, together with the temperature, must be so handled as to embrace the general and local circulation in a single comprehensive system.

Such are the comments of a man after spending more than 15 years arduous study to the subject of theoretical meteorology.

It seems quite certain, then, we are dealing here with a very fundamental and important difficulty, if not a real error in the dynamics of the atmosphere, due apparently to the treatment of, or limitations which have been imposed upon, the general equations of motions by their several authors.

Rational polar cyclone.

Let us see if it is not possible to construct a more rational answer to the crucial question of the circumpolar cyclone by applying to the problem the equations for cyclonic frictionless or gradient winds.

For this purpose we will confine ourselves closely to results drawn directly from observations. Figure 7 shows the mean January pressure over the Northern Hemisphere at an elevation of 1,500 meters, as redrawn by Bigelow [20] from studies by Tiesserenc de Bort.

Following the mathematicians' assumptions that temperature is constant along all parallels but progressively different from latitude to latitude, there is only one idealized barometric map of such a polar condition possible, namely, a system of concentric circular isotherms and isobars each parallel to lines of latitude. Such a diagram, in no material sense, differs from the cyclone of figure 5, and for simplicity figure 7 is idealized and redrawn in figure 8.

The outermost isobars are shown as fragments of straight lines and as arcs of very great radii because as wind tracks the Equator is a straight line and the adjacent parallels have very great radii of curvature. This helps the mind to see in the diagram the effects resulting from the curvature of the earth from the Equator to the pole.

There is very little *a priori* basis on which we can say definitely what the actual pressures should be from latitude to latitude in this hypothetical case, but the observations indicate a tendency to the existence in the lower strata of a permanent high-pressure belt along tropical latitudes accompanied by low pressures at the Equator and the poles. This is very clearly apparent in de Bort's map. In an arbitrary manner, therefore, let the belt of maximum pressure be represented by the very heavy isobar in figure 8 at latitude about 30°. Within this isobar the circulation must necessarily be cyclonic because the gradient decreases toward the center. Between this isobar and the Equator the circu-

[19] A contribution to this question of far-reaching importance has been made recently by G. I. Taylor on eddy motions in the atmosphere. See Phil. Trans., vol. 215, 1915, pp. 1-26. Proc. Roy. Soc., Ser. A, vol. 92, pp. 196-199.
For abstract and review of above by Eric R. Miller, see MONTHLY WEATHER REVIEW, vol. 47, October, 1919, p. 703. C. F. M.

[20] Bigelow: International Cloud Report, loc. cit. Tiesserenc de Bort: Report on General Circulation of the Atmosphere, London, 1893.

lation is obviously anticyclonic. For frictionless motions the velocities are given by equations (15a) and (17a), but if we disregard, as is fully justified, a region near the poles the actual *curvature* of the remaining isobars can everywhere be entirely neglected with scarcely appreciable error. Consequently equation (13) for straight isobars gives the desired relation between the wind velocities, latitude, and pressure gradients. Now we can not know exactly what the final permanent pressure and temperature gradients would be after the atmosphere had attained a stable state of perfectly frictionless motion. We do know, however, that the friction in the free air, at least, is very small on the whole and we are compelled to conclude that until time permits temperature contrasts to change the pressure gradients as shown in figure 7 very nearly at least represent gradients for frictionless flow. With friction the winds must necessarily blow

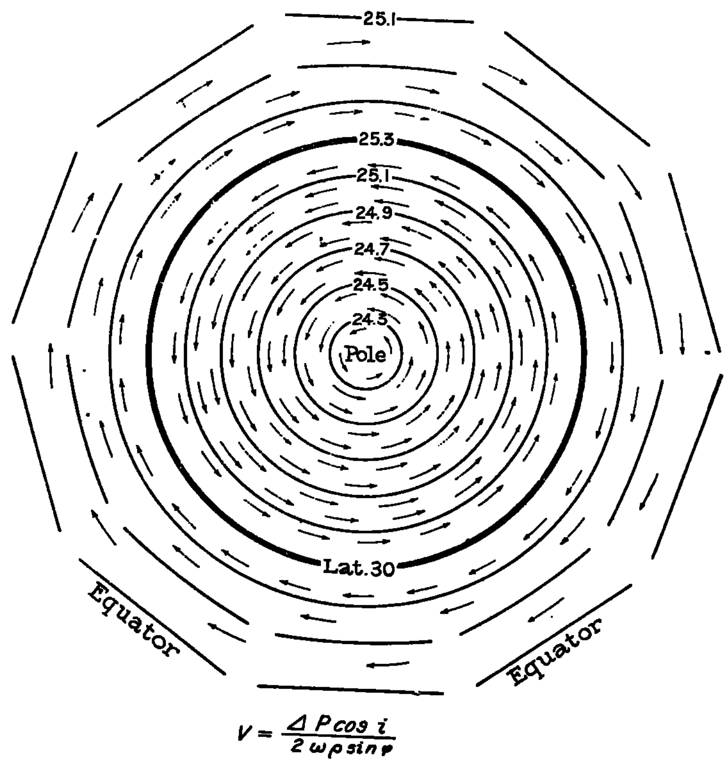

FIG. 8.—Figure 7 idealized by assuming pressure constant along parallels of latitude and belt of high pressure uniformly at latitude about 30°. The pressure and gradients are as nearly as practicable the same as in figure 7.

across the isobars at a certain angle i, and the velocity will then be given by equation (12), viz:

$$V = \frac{\delta p \cos i}{2 \rho \omega \sin \phi} \qquad (12)$$

Now the angle i is everywhere quite small except close to the surface of the earth and near the Equator. Even if i has a value of 10° or 15° the value of cosine i will differ but 1 or 2 per cent from unity, hence perfectly frictionless winds under these conditions can not exceed the velocities *with* friction by more than a very few per cent. These considerations which set forth the difference between winds with and without friction are too generally overlooked and disregarded. The important point is that *steady winds flow across the isobars at angles as great as 10° to 15° with only a small falling off in velocity below the gradient velocity.*

Assuming figure 8 to represent literally frictionless conditions we can easily calculate the gradient velocities for the different latitudes from equation (13). For this purpose we must recognize that over the belt of maximum pressure the gradient and hence the velocity is zero. Toward the Equator it is feeble, as shown by Teisserenc de Bort's chart, viz., about 4 mm. in 30°, which is a gradient of only one-tenth inch per 1,400 miles. Within the heavy isobar the gradients are steeper and nearly uniform except that data are mostly wanting within 10° or 15° of the pole. We know to a certainty, however, that no unusual gradients obtain even here. The data we have show we are fully justified in spacing the polar isobars at about one-tenth inch per 5°, or 350 miles. Taking $\rho = .00100$, equation (13), also Smithsonian Meteorological Tables, 4th edition, Table No. 42, gives the approximate gradient winds for this *rational circumpolar cyclone* as shown in the following Table 3, in which is also included the velocities for the circumpolar frictionless hurricane arrived at by the mathematicians.

TABLE 3.—*Contrasting the frictionless circumpolar winds in the lower strata derived by the mathematicians and the gradient winds in a rational circumpolar cyclone.*

[Velocities in miles per hour. +eastward, −westward flow.]

Latitude	Mathematical cyclone velocity.	Rational cyclone velocity.	Assumed gradient: $\frac{1}{10}$ inch per—
° ′			*Miles.*
0 00	− 346	−00	1,500
10 00	− 320	−12	1,500
20 00	− 239	− 6	1,500
30 00	− 100	− 0	∝
35 15	0	+ 6	1,000
40 00	+ 108	+16	350
50 00	+ 410	+13	350
60 00	+ 865	+12	350
70 00	+1,669	+11	350
80 00	+3,807	+10	350
90 00	+ ∝	?	?

Obviously, the velocities by the two methods of solution are irreconcilable to a striking extent, although both in fact are derived from exactly the same fundamental dynamic principles. The analytic treatment in one case leads to unknown and inconceivable winds at the poles and excessive velocities at the equator, while the results reached by the other treatment are rational throughout.

The writer is convinced the gradient winds are correctly derived, whereas the mathematicians in the final solution have circumscribed and limited their general equations in a manner which has literally precluded the operation of the law of the geoidal slope and has left the law of equal areas free to require the inconceivable velocities to which their equations lead. This is obviously a difficulty only in the *solution*, not in the original *statement* of the equations. The one solution which seemingly all mathematicians have been satisfied to follow is fallacious because the latitude effect and the slope effect are not simultaneously satisfied, as must be the case.

The writer particularly requests that the reader will not suppose that the so-called *rational polar cyclone* is offered as a representation of what exists in nature. Nothing of the kind is intended. It is a mere fragment and is offered to show a *method of solution* of certain equations of motion in such a way as to satisfy the requirements of terrestrial mechanics. In this method the *gradient* is recognized and treated as the independent variable. Moreover, the dependent deflective influences due to relative velocity and the earth's rotation are simultaneously, not separately and independently, satisfied as in the usual mathematical treatment. To make the application of this method to the entire polar hemisphere, isobaric maps must be available for all altitudes from the surface to the limits of the atmosphere. More attention must be given to the motions *in the vertical.* The flow of air in any given layer or stratum will be determined by the pressure distribution *in that stratum*, and this can not be correctly inferred in many cases from the surface pressures, as meteorologists are too prone to do.

There is no inherent necessity whatever that the ideal circumpolar cyclone shall be centered at the pole. The center might just as well be over Alaska or Iceland or elsewhere. There may be even two or more centers. The circulation is produced *by the gradient*, which in turn results from temperature contrasts. The rotation of the earth simply requires a gyratory circulation, whereas the mathematicians have fairly made the *rotation* produce the circulation. This seems to result from a failure to discriminate adequately between *real active forces*, like a pressure gradient, and those other things, like the momentum effect, the slope effect, the earth's deflective influence, and the so-called centrifugal force, which are all passive inertia reactions and which unfortunately are treated too often as if they were real forces capable in and of themselves of *causing* definite motions. All these quantities, including friction, are wholly *dependent* terms in the physics of these questions. It will be noticed that the local and the circumpolar cyclone are identical mathematical concepts except for the following unimportant particulars.

The polar cyclone is large, with many isobars of enormous radii of curvature, is cold at the center and warm at the outer limits. The value of f, the deflective influence of the earth's rotation, varies all the way from zero at the equator to a maximum finite value at the poles. Finally, the polar cyclone is stationary.

The local cyclone is small, with isobars of short radii. The contrasts of temperature in it are uncertain and probably of incidental significance. The value of f changes but little throughout its extent. Finally, in general the system often has a definite motion of translation.

Nonrotating earth.—It will also be noticed that a correct mathematical theory of the cyclone on a rotating earth can be transferred bodily to the atmosphere of a stationary earth, and vice versa, assuming of course all other conditions remain unchanged. For the stationary earth, $f=0$, and some initial impulse is then required to start a gyration, which may be clockwise or counterclockwise *ad libitum.* In other words, the only effect of the earth's rotation on the mathematical theory of cyclones is (*a*) to introduce into the equations of motion the passive reaction f, of variable magnitude, depending upon latitude and velocity; (*b*) to compel gyration in all cases and always in a particular direction. These considerations prompt the suggestion to the mathematicians that the theory of cyclones be first built up for conditions of a stationary earth.

Reference to the diagram of forces, figure 5, for the anticyclone shows that if $f=0$ (stationary earth) there can be no force to give curvature to the wind tracks. *Hence what we now know as anticyclonic circulation could have no existence on a stationary earth.* All gyratory circulation must necessarily be cyclonic on a stationary earth, just as is found to be the case on the rotating earth near the Equator, where f is small or zero. However, on a stationary earth gyration must be set up by some extraneous force and may be in either direction.

There is another very important consideration I wish to urge.

Ferrel's generalized frictionless polar cyclone with no change of temperature or pressure in longitude, as likewise the idealized rational cyclone of figure 8, even any system of mean monthly isobars, as Tiesserenc de Bort's chart for January, should not be supposed to REPRESENT a polar circulation and pressure. The most that can be said of these generalized, averaged, or ideal systems is that they lead to a circulation and pressure distribution that are *equivalent* to the actual circulation. By actual I mean the circulation hour by hour, day by day, with all its great changes.

We may say that a triangle is *equivalent* to a circle if the areas are equal, but the one form can not represent the other. Neither does the generalized circulation *represent* what actually occurs in nature, although the final effects may be equivalent the one to the other.

In the idealized polar cyclone the isobars necessarily run parallel to the latitude and for frictionless gradient motion interpolar flow is impossible. Actually, however, the isobars from hour to hour and day to day cross the lines of latitude *ad libitum* with unrestricted interpolar flow possible at any and all times with or without friction.

Critical mathematical examination may show it to be impossible to construct an *idealized* circumpolar circulation which is equivalent to and fairly represents the actual daily circulation *and at the same time will fully satisfy the requirements of the law of equal areas and friction as they exist.* Such an idealized circulation can not be an actual circulation but only equivalent thereto.

We must not, therefore, assume, although it seems perfectly plausible to do so, that the pressure and temperature are constant along lines of latitude. This limitation involves an incompatability with the operation of the law of equal areas and leads to irrational results because it precludes isobaric lines from crossing parallels of latitude as we find them to do freely in nature.

If we let friction modify as it does the motions in the ideal cyclone of figure 8, what are the results? The difference is very slight. Let the friction angle i be 10°. In the free air it is probably less. By equation (12) the "steady" winds will fall only $1\frac{1}{2}$ per cent in velocity below the gradient velocity of Table 3, and by the equation

$$\text{Motion along meridians} = V \sin i$$

we find the whole mass of air could move poleward at a velocity of about 17 per cent of the actual linear motion. This method of analysis, although obviously incomplete in many details, nevertheless leads to a perfectly moderate and rational cyclonic circulation either *with or without* friction.

Returning to a consideration of Table 3, the remarkable feature of the gradient winds therein is the very moderate velocities everywhere except near the Equator, where alone the theoretical winds are seemingly inconsistent with observation. Any of these values will be halved or doubled by doubling or halving the spacing of the isobars. Clearly, therefore, these gradient winds harmonize in a wholly satisfactory manner with observations, because even in the equatorial belt of calms the discrepancy is not real, but only apparent, for two reasons: (1) The gradients here in general are very feeble, less than one-tenth inch per 1,000 miles. (2) The circulation in many cases, *probably never* attains any approach to the case of gradient winds, especially at the surface. Feeble gradients, slight temperature contracts, and weak deflective forces over a wide equatorial belt constitute conditions under which the air flows easily and directly from place to place at small velocities, which even very slight friction suffices to hold down far below the theoretical velocities representing frictionless motions under strictly gradient conditions.

It is very certain from the foregoing that the equations of the mathematicians for frictionless motion on a rotating globe must yield wind velocities comparable with those in Table 3 for gradient winds before one can be satisfied that a fairly satisfactory analytical solution of the general circulation of the globe is attained. Possibly the problem may be approached anew by recognizing that the great systems of permanent isobaric lines for the surface and free air really mark out the great winding lanes and highways nearly parallel to which the perpetual flow and counterflow of the air must take place between the equator and the poles.

It is impossible in this already overlong article to cover fully all phases of the issues raised or point out the numerous fallacies in textbooks consequent upon the general disregard of the action of the slope effect in atmospheric motions.

Hann and Hildebrandsson,[21] with others, have questioned the accuracy of Ferrel's theories of the general circulation by showing material inconsistencies between theory and observations. The writer does not know, however, that any serious question has been raised heretofore regarding the soundness of the mathematical analysis itself or the correctness of the application of the physical and dynamic principles involved. If the representations in this paper are sustained, the cause for discrepancies between theory and observations is identified.

A very convincing proof that the claim of excessive polar winds for frictionless conditions is erroneous may be found in the sober reflection that atmospheric friction, especially in the free air, is very small at the most and that many observations tell us polar winds and pressure gradients do not differ materially from winds and gradients elsewhere.

Pilot balloon observations.—The foregoing studies give a new significance and importance to the observations in the free air by means of pilot balloons, which in a comparatively inexpensive way tell us at least during clear weather the velocity and direction of the wind in the various strata. By extending the observations to the highest altitudes attainable and occupying stations in equatorial and polar regions, about which little definite is now known, information of great value upon the motions and pressure gradients in the free air can be secured.

Obviously much work is now necessary to reconstruct a new mathematical analysis of atmospheric motions free from the faults it is believed have been brought to light in this paper.

SUMMARY AND CONCLUSIONS.

The writer is conscious the foregoing representations are more or less involved and indirect because it has been necessary, first, to establish and call attention to certain deep-seated and widespread errors in meteorological literature and at the same time to clearly explain the operation of certain obscure dynamic actions which, while heretofore known, nevertheless have been neglected or not consistently applied in mathematical writings as they should have been. It, therefore, appears appropriate to conclude this paper with a number of categorical statements giving important principles which the motions of the atmosphere must satisfy.

I. The rotation of the earth or any globe on its axis gives rise to two inseparable, independent, dynamic reactions upon matter in free relative motion. (1) The conservation of angular momentum requires that changes of latitude be accompanied by changes of velocity in longitude. (2) A component of centrifugal reaction introduces forces in the plane of the meridians whenever there are any relative motions in longitude. (3) These two reactions must always be simultaneously satisfied. Their resultant is the so-called deflective influence of the earth's rotation, which is entirely passive in its effects.

Free frictionless motions.—A body set into free frictionless motion over a smooth rotating globe by some initial force which then ceases to act will continue in motion forever, never changing its velocity, but constantly changing the direction of the motion unless it is exactly along the equator.

II. *Sustained force and ultimate velocity.*—If the force is not of a periodic character but sustained indefinitely a *constant velocity of finite value* will be attained. Enormous or infinite velocities result only from enormous or infinite forces. An initial impulse imposed upon an atmosphere at rest leads in general to initial surgings and oscillations with subsidence into eddy motions and vortical secondary developments which ultimately must embrace the whole atmosphere if not dissipated by friction. A permanent circumpolar cyclone, even if assumed to be limited to one hemisphere, can not be induced by an *elementary* initial impulse, as Ferrel assumed. The initial cause must be adequate to create the cyclone which would then continue forever if not subsequently dissipated by friction.

III. *Deflective influence passive.*—Neither the deflective influence not its components can act alone and *produce* motions. Their parts are to control and modify the directions *only*, of motions whenever set up or maintained by extraneous forces.

IV. *Active forces and velocities attainable.*—Since the deflective influences are powerless to produce or change velocities, therefore the velocities which can be attained will depend entirely upon the extraneous forces and the resistances encountered by the motions thus set up.

V. *Deflective influences always satisfied.*—Whatever entirely free motions may be set up, the demands of the slope effect and the latitude effect will always be automatically satisfied according to the momentary velocity in longitude and the changes in latitude. If the motions are constrained, as by tracks, banks or like fixed obstacles, the demands of the deflective influences will be expressed as lateral pressures of some kind.

VI. *Friction always present to produce rest.*—In all atmospheric motions, convections, turbulence, frictions and internal wastes of energy of many kinds are always present. These will soon dissipate and stop any free motions not continually maintained by active forces.

VII. *Pressure gradients active forces.*—The immediate forces which produce the general motions of the atmosphere arise by virtue of, and are measured by, pressure gradients.

VIII. *Gravity, pressure gradients, temperature contrasts.*—Pressure gradients in air and other fluids are *gravity reactions* and in the grand case of the whole atmosphere the permanent gradients depend chiefly upon the great *contrasts of temperature* which arise and are perpetually maintained by the unequal heating of the earth's surface by the sun which, in conjunction with the continuous losses of heat by terrestrial radiation,

[21] Hann: MONTHLY WEATHER REVIEW, 1914, vol. 42, p. 612. Hildebrandsson: MONTHLY WEATHER REVIEW, 1919, vol. 47, p. 374.

cause the perpetual warmth of the tropics and the extreme cold of the polar regions.

IX. *Steady motions under balanced forces.*—Whatever free motions may be continuously maintained against resistances by active forces, a state of steady motion under balanced forces must soon be established. If the active forces and the opposing resistances are constant the velocities in the steady state will always be constant regardless of the direction of motion. Changes of both velocity and direction must always accompany changes in amount or character of the active forces or resistances, except in the improbable case in which simultaneous changes in both force and friction just offset each other.

X. *Flow of air tends to minimum, or state of rest.*—The flow of masses of the air from places of higher to places of lower pressure obviously at once tends to reduce or dissipate the pressure gradient to fill up the low, after which friction stops the motion. Such flow also tends to reduce or remove temperature contrasts and any like causes which tend to create and maintain gradients. In other words, *all motions of the atmosphere due to temperature contrasts and pressure gradients tend automatically to the minimum of motions, or to a state of rest.* First, because even without friction the flow and intermixture must equalize temperatures and dissipate gradients or reduce these to a minimum. Second, the surging and oscillating motions of great complexity which conceivably might be set up and continue forever without friction, must by it be readily damped out or reduced to a steady state of the minimum motion.

XI. The motions of the air must satisfy the equations of continuity which require that the inflow and outflow for a given region shall be equal on the average.

XII. Hadley and others since assume, without adequake basis of proof, however, that the algebraic sum of all the frictional affects between the air and the earth for the entire surface of the globe is zero, because otherwise a change in the period of rotation of the earth on its axis should be in evidence.

The foregoing is believed to clearly state principles of great fundamental importance in dynamic meteorology. The whole difficult problem of the mechanics and thermodynamics of the atmosphere is comprehended in the steady winds and the changing motions, for which simply the conditions are stated in paragraphs IX and X.

The literature of mathematical meteorology in so far as it relates to atmospheric circulation is an effort to satisfy in mathematical terms principles IX and X.

THE GREAT CYCLONE OF MID-FEBRUARY, 1919.

By C. LEROY MEISINGER.

[Weather Bureau, Washington, D. C., Oct. 23, 1920.]

SYNOPSIS.

Between the 10th and 16th of February, 1919, the United States witnessed the passage of a cyclonic storm of more than usual intensity, with almost circular isobars, with a diameter sufficient to overreach the northern and southern borders of the country, and with a persistence which enabled it to retain its identity from the time it appeared in the western United States until it disappeared off Newfoundland. A study of this storm, based upon the upper-air data obtained at stations of the Meteorological Section of the Signal Corps and of the Weather Bureau, shows that the distribution of weather elements agrees closely with the usual conditions as described by Bjerknes. The influence of the storm extended at least as high as 3 kilometers, as shown by kites and pilot balloons. There were high wind velocities, both at the surface and aloft, which gave rise to widespread dust storms in the Middle West. Eight maps show the distribution of pressure and winds (surface and aloft) from the 12th to the 15th, inclusive.

INTRODUCTION.

It is not often that the endless procession of low-pressure areas, sweeping across the United States from west to east, reveals a member so strikingly symmetrical, so intense, so persistent, and so remarkable in the distribution of cloudiness and precipitation as that of February 10 to 16, 1919. Appearing on the morning of February 10 off the coast of British Columbia, it moved southeastward into the United States, and by the morning of the 11th was centered in northern Nevada. The morning of the 12th found it centered at Denver, the 13th in eastern Kansas, the 14th in central Illinois, the 15th in New England, and the 16th found it over the Atlantic east of Newfoundland. Its greatest intensity was observed on the morning of the 13th at Kansas City, where the sea-level pressure was 28.90 inches. Previous to this time it had been gradually deepening, and in the remainder of its journey across the United States it diminished very slowly until it approached the ocean, when it appeared to intensify slightly because of the warmer air, the lesser surface friction over the water, and the increasing latitude. With such a strong horizontal gradient of pressure, it was natural that there should be high winds; and with such active circulation that there should be a strong surface temperature gradient. Therefore, because of the almost ideal characteristics of this LOW, it has seemed worthy of study, not only as to surface weather, but also as to the winds aloft.

WEATHER AND WINDS.

Precipitation.—In several recent papers [1] on the general subject of forecasting weather, Prof. V. Bjerknes has outlined in a very lucid manner the way in which masses of cold and warm air interact in circulating about a barometric depression, based upon his observations in Norway. There are two distinct lines of discontinuity in the moving cyclone, *the steering line*, which is shaped like an inverted "S" and occurs in the eastern half of the depression, marking the front, at the surface, of a tongue of warm southerly air; and *the squall line*, which trails away from the center into the southwest quadrant of the cyclone, marking the rear of the intruding tongue of warm southerly air. Along the steering line the southerly air leaves the surface and overrides the easterly surface current in the northern part of the depression; along the squall line the cold northerly wind of the western side of the depression underruns the tongue of warm southerly air. Precipitation is closely related to these two lines: along the steering line the rain falls owing to the dynamic cooling of the southerly wind as it rises over the easterly, and along the squall line the rain falls from the southerly air, which is forced to ascend by the denser northerly wind. A third cause of rain is frequently operative also, namely, the convection caused by the convergence of winds within the tongue of southerly air.

The reason for outlining thus the observations of Prof. Bjerknes is to draw attention to the striking accord which exists in the performance of the cyclone in question. From the time the storm freed itself from the topographical hindrances of the Rocky Mountains the distribution of winds and precipitation during its eastward march conformed perfectly with the mechanical outline of Bjerknes. On the 12th, when the storm was centered

[1] The structure of the atmosphere when rain is falling. *Quar. Jour. Royal Meteorological Society*, April, 1920, pp. 119-140; abstract in MO. WEATHER REV., July, 1920, 48:401. The meteorology of the temperate zone and the general atmospheric circulation. *Nature* (London), June 24, 1920, pp. 522-524; abstract in later REVIEW.